# 5 Implication-Space Semantics

## Content as Implicational Role

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In the previous chapter, we have seen that the pragmatic-normative understanding of open reason relations from Chapters One to Three is isomorphic to the semantic-representationalist understanding of open reason relations in terms of truth-makers and falsity-makers. In particular, if collections of assertions and denials of atomic sentences are normatively ruled out just in case all fusions of truth-makers of the asserted sentences and falsity-makers of the denied sentences are alethically ruled out, then the pragmatic-normative theory of consequence among sentences from Chapter Three coincides with the semantic-representationalist theory of consequence among worldly propositions from Chapter Four. That is, the consequence relation among sentences defined in pragmatic-normative terms and the consequence relation among worldly propositions, that are represented by these sentences, defined i n semantic-representationalist terms are isomorphic under the mapping of representations to representanda. We thus have a common formal structure of open reason relations that can be explicated in a pragmatic-normative and also in a semantic-representationalist metavocabulary. We called the roles in this formal structure that are shared between sentences and worldly propositions their "rational forms."

In this chapter, we present a theory of these rational forms in themselves and, hence, not as they occur in discursive acts or in worldly states. We thereby abstract away from the different kinds of matter in which rational forms can occur. We present an intrinsic metavocabulary of reason relations, a metavocabulary that allows us to talk about the content of something as its role in reason relations: its role in implications. In other words, we present an account that does not appeal to anything like pragmatic-normative or semantic-representationalist ideas to explain reason relations, but provides rather a metalinguistic codification and explication of reason relations that appeals only to the reason relations of the base vocabularies themselves. We call this theory "implication-space semantics."[1] From the perspective of implication-space semantics, the pragmatic-normative and the semantic-representationalist theories from previous chapters can be seen as accounts of two particular ways in which rational forms, that is, implicational roles, can be realized.

The chapter is organized as follows. In the first section, we introduce implication-space semantics. We then explain how the pragmatic-normative account in terms of the sequent calculus NMMS and the semantic-representationalist account in terms of truth-makers can be interpreted in implication-space semantics. In the third section, we explain how implication-space semantics reveals the basic structure of reason relations to be that of a monoid defined on a set of pairs, together with a subset of the monoid set. Section 5.4 shows how implication-space semantics can capture non-contractive reason relations and multiplicative additive linear logic. We then present, in Section 5.5, an account of relations among implicational roles, which allows us to recover logics like the logic of paradox (LP) and strong Kleene logic (K3). Section 5.6 concludes.

## 5.1 Formulating Implication-Space Semantics

The extrinsic metavocabularies of norms governing acceptance and rejection and of states making sentences true or false, which were our topics in the previous two chapters, appealed to something outside of reason relations themselves in order to explain what it means for things to stand in reason relations. We now abstract away from such appeals to external resources and formulate a theory that explicates reason relations merely in terms of the materials already available in reason relations themselves. The theory provides a notion of conceptual contents as roles within reason relations. These conceptual roles are the rational forms shared by contentful sentences and worldly propositions, which we already encountered in the previous chapter.

### *5.1.1 Implicational Roles*

In the previous chapter, we introduced the idea of rational form as the modal role that something plays in relations of exclusion. These relations of exclusion are the generic form of reason relations, of implications (including incompatibilities).[2] Hence, in order to focus on rational forms, we must look directly at roles that things play in reason relations or implications. We will call these roles "implicational roles." We therefore begin by making the notion of an implicational role precise.

Although we want to focus on rational forms and, hence, on implicational roles in abstraction from anything that embodies these forms or roles, we still need some placeholders for bearers of roles, which we will simply call "bearers." In the two previous chapters, we saw that sentences and worldly propositions can serve as bearers of implicational roles. In this chapter, we abstract away from any particular bearers and focus merely on the roles they play in implications. Our general strategy is to define roles as equivalence classes of bearers or implications among bearers. We will provide a general way to abstract implicational roles from any given set of bearers standing in implication relations, where these implication relations can, crucially, be structurally open reason relations.

We will start by considering an otherwise unspecified set of bearers of implicational roles, and the set of all possible implications among them, which we call "candidate implications." We call the set of all candidate implications the "implication space" of a given set of bearers.

**Definition 61** (Role bearers and implication space)**.** The members of a non-empty set $\mathtt{B}$ are bearers of implicational roles. And the set, $\mathbb{S}$, of all the pairs of subsets of $\mathtt{B}$ is the bearer implication space, which contains all the candidate implications among the bearers, so $\mathbb{S} = \mathcal{P}(\mathtt{B}) \times \mathcal{P}(\mathtt{B})$.

We call the first set in a candidate implication its "premises" and the second its "conclusions." Some candidate implications hold and others do not hold, that is, some candidate implications are good implications and some are not. Thus, the actual implications or good implications are a subset of the implication space. We use "implication frame" for a pair of the set of bearers and a subset of good implications.

**Definition 62** (Implication and implication frames)**.** Implication is a relation between sets of bearers, $\mathbb{I} \subseteq \mathbb{S}$. An implication frame is a pair $\langle \mathtt{B}, \mathbb{I} \rangle$ of set of bearers, $\mathtt{B}$, and implications among sets of them, $\mathbb{I} \subseteq \mathbb{S}$.

The idea behind this definition is that an open reason relation holds among sets of bearers and that $\Gamma \mathrel{|\!\sim} \Delta$ is part of this reason relation just in case $\langle \Gamma, \Delta \rangle \in \mathbb{I}$. We will sometimes call the minimal element, $\langle \emptyset, \emptyset \rangle$, of the implication space $\mathtt{e}$, and we denote the maximal element $\langle \mathtt{B}, \mathtt{B} \rangle$ by $\star$. We assume that $\mathtt{e} \notin \mathbb{I}$ but $\star \in \mathbb{I}$.

We want to define the roles that bearers play in implication relations, and these roles should be such that it is possible for distinct bearers to play the same role. To a first approximation, we can think of an implicational role as represented by the collection of bearers that all play the same role. This suggests that we start by asking when two bearers play the same role. Our strategy in this section is to first define an equivalence relation between bearers, and indeed implications and sets thereof, which holds just in case the equivalent items play the same implicational role. And we then define implicational roles as equivalence classes with respect to this equivalence relation.

In order to define this equivalence relation, consider that, intuitively, the implicational role of a bearer has two parts: firstly, the role that the bearer plays as a premise in implications and, secondly, the role that it plays as a conclusion in implications. We can think of the role that a bearer plays as a premise or a conclusion as the contribution that it makes to the goodness of an implication, as a premise or as a conclusion, respectively. In other words, if we have a bearer, $\phi$, of an implicational role, then the role of $\phi$ as a premise in implications is fully settled by for exactly which sets $X$ and $Y$ we have $\langle X \cup \{\phi\}, Y\rangle \in \mathbb{I}$, that is, by exactly which good implications the bearer figures in as a premise. And the role of $\phi$ as a conclusion in implications is fully settled by for exactly which sets $X$ and $Y$ we have $\langle X, Y \cup \{\phi\}\rangle \in \mathbb{I}$, that is, by exactly which good implications the bearer figures in as a conclusion. We call the first part of this role of a bearer its "premisory role" and the second part its "conclusory role." The premisory role of a bearer tells us what contribution the bearer makes to good implications as a premise. And the conclusory role of a bearer tells us what contribution the bearer makes to good implications as a conclusion.

Considering premisory and conclusory roles naturally leads to a broadening of our focus from roles of particular bearers to roles of whole implications. To see this, it is helpful to recall the definition of the range of subjunctive robustness from Chapter Three. The idea behind ranges of subjunctive robustness is that, especially in a nonmonotonic setting, it is interesting to consider which additions to an implication do not defeat the implication, if the implication is already good, or turn the implication into a good implication, if it is not already good. If we generalize that definition from $\mid\!\sim$ to $\mathbb{I}$ and also extend the definition to sets of candidate implications, the result is the following:

**Definition 63** (Range of subjunctive robustness, $\mathtt{RSR}(\cdot)$)**.** Given $\langle \Gamma, \Delta\rangle \in \mathbb{S}$, its range of subjunctive robustness, $\mathtt{RSR}(\langle \Gamma, \Delta\rangle)$, is the set of pairs, $\langle X, Y\rangle$, such that $\langle \Gamma \cup X, \Delta \cup Y\rangle \in \mathbb{I}$; that is, $\mathtt{RSR}(\langle \Gamma, \Delta\rangle) = \{\langle X, Y\rangle \in \mathbb{S} \mid \langle \Gamma \cup X, \Delta \cup Y\rangle \in \mathbb{I}\}$. The range of subjunctive robustness of a set of candidate implications is the intersection of the ranges of subjunctive robustness of the members; that is, if $H \subseteq \mathbb{S}$, then $\mathtt{RSR}(H) = \{\langle X, Y\rangle \mid \forall \langle \Gamma, \Delta\rangle \in H\,(\langle \Gamma \cup X, \Delta \cup Y\rangle \in \mathbb{I})\}$.

Notice that the set of pairs $\langle X, Y\rangle$ such that $\langle X \cup \{\phi\}, Y\rangle \in \mathbb{I}$ is the range of subjunctive robustness of $\langle \{\phi\}, \emptyset\rangle$. And the set of pairs $\langle X, Y\rangle$ such that $\langle X, Y \cup \{\phi\}\rangle \in \mathbb{I}$ is the range of subjunctive robustness of $\langle \emptyset, \{\phi\}\rangle$.[3] So it follows from what we said above that these ranges of subjunctive robustness fully determine the premisory and conclusory roles of a given bearer $\phi$. This suggests that everything with the same range of subjunctive robustness, as a premise and as a conclusion, plays the same implicational role as the given bearer. Thus, if the two bearers $\phi$ and $\psi$ are such that the ranges of subjunctive robustness of their occurrences as a premise are identical, that is, if $\mathtt{RSR}\,\langle\{\phi\},\emptyset\rangle = \mathtt{RSR}\,\langle\{\psi\},\emptyset\rangle$,[4] then $\phi$ and $\psi$ play the same premisory role. And if the ranges of subjunctive robustness of their occurrences as a conclusion are identical, that is, if $\mathtt{RSR}\,\langle\emptyset,\{\phi\}\rangle = \mathtt{RSR}\,\langle\emptyset,\{\psi\}\rangle$, then $\phi$ and $\psi$ play the same conclusory role.

To formulate this idea in a rigorous way, let us define a notion of equivalence among sets of candidate implications in terms of their ranges of subjunctive robustness. Individual candidate implications, such as $\langle\{\phi\},\emptyset\rangle$ and $\langle\emptyset,\{\phi\}\rangle$, can then be seen as the special case where the sets of candidate implications have only one member.

**Definition 64** (Implication equivalence, $\approx$)**.** Let $G$ and $F$ be sets of candidate implications, then $G \approx F$ if and only if $\mathtt{RSR}(G) = \mathtt{RSR}(F)$.

Since it is defined by an identity, this relation is obviously symmetric, reflexive, and transitive. So implication equivalence is indeed an equivalence relation. We can now consider classes of sets of implications that are equivalent to one another, in the sense of implication equivalence. Indeed, we can define implicational roles as such equivalence classes of candidate implications. Given that we treat the roles of bearers in terms of premisory and conclusory roles and we have defined equivalence not only for implications but for sets of implications, we can define implicational roles not only for bearers but also for implications and sets of implications, namely as follows:

**Definition 65** (Implicational Role, $\mathcal{R}(\alpha)$)**.** The implicational role, $\mathcal{R}$, of something, $\alpha$, is its equivalence class under $\approx$, namely[5]:

(1) If $\alpha \subseteq \mathtt{S}$, then $\mathcal{R}(\alpha) = \{x \mid \alpha \approx x\}$.
(2) If $\alpha \in \mathtt{S}$, then $\mathcal{R}(\alpha) = \mathcal{R}(\{\alpha\})$.
(3) If $\alpha \in \mathtt{B}$, then $\mathcal{R}(\alpha) = \langle\mathcal{R}^+(\alpha), \mathcal{R}^-(\alpha)\rangle$, where $\mathcal{R}^+(\alpha) = \mathcal{R}\,\langle\{\alpha\},\emptyset\rangle$ and $\mathcal{R}^-(\alpha) = \mathcal{R}\,\langle\emptyset,\{\alpha\}\rangle$.

Clause (3) of this definition says that the implicational role of a bearer, $\alpha$, is the pair of the implicational role of $\langle\{\alpha\},\emptyset\rangle$ and the implicational role of $\langle\emptyset,\{\alpha\}\rangle$, as defined in clause (2) of the definition. These roles of implications, in turn, are equivalence classes of implications, namely sets of implications that have the same range of subjunctive robustness. Clause (1) gives the general case that defines roles of sets of candidate implications, namely as equivalence classes of such sets.

To illustrate, the premisory role of bearer $\phi$ is the set of sets of candidate implications with the same range of subjunctive robustness as $\langle\{\phi\},\emptyset\rangle$. For example, the singleton of a candidate implication $\{\langle\Gamma,\Delta\rangle\}$ is in this set if and only if parallel additions of premises or conclusions to $\langle\Gamma,\Delta\rangle$ and to $\langle\{\phi\},\emptyset\rangle$ always either both yield good implications or both yield implications that are not good. In other words, in any candidate implication, we can replace the bearer $\phi$ as a premise *salva consequentia*—that is, without turning a good implication into a bad one or vice versa—with the combination of $\Gamma$ as premises and $\Delta$ as conclusions, and the other way around. So, if two bearers have the same premisory role, then they can be substituted for each other as premises *salva consequentia*. Conclusory roles are analogous, except that $\langle\{\phi\},\emptyset\rangle$ is changed to $\langle\emptyset,\{\phi\}\rangle$. Hence, if two bearers have the same conclusory role, they can be substituted for each other as conclusions *salva consequentia*. The equivalence classes with respect to ranges of subjunctive robustness that are implicational roles thus capture the idea of "playing the same role in implications."[6]

According to the notion of an implicational role that we have just introduced, what it means to play a particular implicational role is to be a member of a particular equivalence class of things with the same range of subjunctive robustness. For the case of individual bearers of implicational roles, their roles are pairs of such equivalence classes, namely the premisory and the conclusory roles of the bearer.

### *5.1.2 Conceptual Content and Entailment*

An implicational role is an abstraction from the implications among bearers of implicational roles. Two distinct bearers can have the same implicational role. And, there can be roles that are not played by any bearer.[7] For there may be ranges of subjunctive robustness for which there is no particular bearer whose premisory or conclusory role is the equivalence class of that range of subjunctive robustness. With implicational roles, we have entered a realm of abstract entities that we may call "conceptual contents."

What we mean by "conceptual content"—or just "content" for short—is an implicational role that a bearer could play. Thus, a conceptual content is a pair of a premisory role and a conclusory role.

**Definition 66** (Conceptual Content). If there are two implicational roles $\mathtt{a}^+ = \mathcal{R}(F)$ and $\mathtt{a}^- = \mathcal{R}(G)$ (for some sets of candidate implications $F$ and $G$), then the pair of them is a conceptual content, $\mathtt{a}$. That is, $\mathtt{a} = \langle\mathtt{a}^+,\mathtt{a}^-\rangle$, where $\mathtt{a}^+$ is the premisory role and $\mathtt{a}^-$ is the conclusory role of the content. The collection of all contents is called $\mathbb{C}$. (Convention: We will use lowercase typewriter font, as in $\mathtt{a},\mathtt{b},\mathtt{c},\ldots$, for conceptual contents and uppercase typewriter font for sets of contents.)

Notice that two bearers of implicational roles from a given implication space have the same conceptual content if and only if one can always substitute one bearer for the other *salva consequentia*. That is, replacing one of the two bearers by the other as a premise or as a conclusion never turns a good implication into a bad one. We might compare this to possible worlds semantics suggesting that two sentences have the same content just in case they can be substituted for each other *salva veritate* with respect to every possible world, that is, in arbitrary modal contexts.

We can now understand implications among bearers, as the concrete manifestation of a more abstract relation of entailment among conceptual contents, although this concrete manifestation is our only access to the more abstract relation. To see what we mean, it is helpful to define the following operation called "adjunction" on implicational roles.

**Definition 67** (Adjunction, $\sqcup$). If $F, G \subseteq \mathbb{S}$, then the adjunction of the implicational roles of $F$ and $G$, written $\mathcal{R}(F) \sqcup \mathcal{R}(G)$, is $\mathcal{R}(\{\langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle \mid \langle \Gamma, \Theta \rangle \in F, \langle \Delta, \Lambda \rangle \in G\})$. We write $\bigsqcup_{i=0}^{n} \mathcal{R}_i$ for the adjunction $\mathcal{R}_0 \sqcup ... \sqcup \mathcal{R}_n$.

Adjunction combines implicational roles by combining by set-theoretic union the premises and the conclusions of the candidate implications that play these roles. This yields a unique result for different members of the implicational roles, so that if $\mathcal{R}(F) = \mathcal{R}(F')$, then $\mathcal{R}(F) \sqcup \mathcal{R}(G) = \mathcal{R}(F') \sqcup \mathcal{R}(G)$ (see Appendix, Proposition 105).

We can use this notion of adjunction to define an entailment relation among contents in terms of implications among bearers. The idea is that the entailment relation holds among sets of contents $\mathtt{G}$ and $\mathtt{D}$ just in case, if we take as premises bearers that play the premisory implicational roles of the contents in $\mathtt{G}$ and we take as conclusions bearers that play the conclusory implicational roles of the contents in $\mathtt{D}$, then this always yields a good implication. More precisely, we say that a set of contents, $\mathtt{G}$, entails a set of contents, $\mathtt{D}$, just in case every set of candidate implications that plays the implicational role of the adjunction of the premisory roles of the elements of $\mathtt{G}$ and the conclusory roles of the elements of $\mathtt{D}$ contains only good implications.

**Definition 68** (Content entailment in an implication frame, $|\!\!\mid\!\!\sim$). Given an implication frame $\langle \mathtt{B}, \mathbb{I} \rangle$, let $\mathtt{G}, \mathtt{D} \subseteq \mathbb{C}$ and $\mathtt{G} = \{\mathtt{g}_0, ..., \mathtt{g}_n\}$ and $\mathtt{D} = \{\mathtt{d}_0, ..., \mathtt{d}_m\}$, then $\mathtt{G} \;|\!\!\mid\!\!\sim\; \mathtt{D}$ holds in the implication frame if and only if $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{g}_i^{+} \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^{-}) \subseteq \mathbb{I}$.

To see what this definition means, let us consider an example. Suppose we have two sets of bearers of implicational roles $\Gamma = \{g_0, ..., g_n\}$ and $\Delta = \{d_0, ..., d_m\}$. Their implicational roles are $\mathcal{R}(g_i) = \langle \mathtt{g}_i^+, \mathtt{g}_i^- \rangle$ and $\mathcal{R}(d_j) = \langle \mathtt{d}_j^+, \mathtt{d}_j^- \rangle$, for all $0 \leq i \leq n$ and $0 \leq j \leq m$. The contents of the bearers in $\Gamma$ entail the contents of the bearers in $\Delta$ if and only if the adjunction of the premisory roles of the bearers in $\Gamma$ and the conclusory roles of the bearers in $\Delta$ has only members that are good implications. In other words, we have to check the adjunction of the following roles: $\mathcal{R}\langle\{g_0\}, \emptyset\rangle = \mathtt{g}_0^+$,..., $\mathcal{R}\langle\{g_n\}, \emptyset\rangle = \mathtt{g}_n^+$, and $\mathcal{R}\langle\emptyset, \{d_0\}\rangle = \mathtt{d}_0^-$, ..., and $\mathcal{R}\langle\emptyset, \{d_m\}\rangle = \mathtt{d}_m^-$. When we perform this adjunction, the result is: $\mathcal{R}\langle\Gamma, \Delta\rangle$. Hence, the entailment holds if and only if every set of candidate implications whose range of subjunctive robustness is $\mathtt{RSR}\langle\Gamma, \Delta\rangle$ includes only good implications.

We have now seen how we can define conceptual content and entailment for any given implication frame, $\langle \mathtt{B}, \mathbb{I}\rangle$.[8] Contents and the entailment relation among them are defined by abstracting away from implications among concrete bearers, in terms of abstract implicational roles.

### *5.1.3 Models and Interpretations*

The notions of content and entailment that we have just introduced can be used to interpret sentences of a language. Recall that we have not specified what bearers of implicational roles are. So, if we are given an implication frame that uses non-linguistic bearers of implicational roles or bearers from one language, we can use the contents defined by that implication frame as interpretants of a(nother) language. As we shall see in this subsection, we can then use such interpretations to give a treatment of logical vocabulary.

In order to give interpretations of logical vocabulary, however, we first introduce another operation on implicational roles, which we call "symjunction." It is another way to combine implicational roles, in addition to adjunction.

**Definition 69** (Symjunction, $\sqcap$). Let $F, G \subseteq \mathbb{S}$, then: $\mathcal{R}(F) \sqcap \mathcal{R}(G) = \mathcal{R}(F \cup G)$.

Similarly to adjunction, the symjunction of roles is unique, in the sense that it does not depend on which bearers we use to pick out the roles whose symjunction we compute. That is, if $\mathcal{R}(F) = \mathcal{R}(F')$, then $\mathcal{R}(F) \sqcap \mathcal{R}(G) = \mathcal{R}(F') \sqcap \mathcal{R}(G)$ (see Appendix, Proposition 106).

Although symjunction is defined in terms of a union of sets of candidate implications, we can think of it as, in a way, isolating the part that two implicational roles share. To see this, note that $\mathcal{R}(F \cup G)$ is the set of sets of candidate implications whose range of subjunctive robustness is $\mathtt{RSR}(F \cup$

$G$). Since we defined the range of subjunctive robustness of a set, however, as the intersection of the ranges of subjunctive robustness of its members, $\mathtt{RSR}(F \cup G)$ is the same as $\bigcap_{x \in F \cup G} \mathtt{RSR}(x)$, which is $\mathtt{RSR}(F) \cap \mathtt{RSR}(G)$. The implicational role of a union of sets of candidate implications is, therefore, the sets of candidate implications that share their range of subjunctive robustness with the intersection of the ranges of subjunctive robustness of the sets of which we take the union. Hence, all of the things that yield a good implication when combined with something that plays the role $\mathcal{R}(F)$, and also when combined with something that plays the role $\mathcal{R}(G)$, yield a good implication when combined with something that plays the role of their symjunction $\mathcal{R}(F) \sqcap \mathcal{R}(G)$. In this sense, as the symbols we have chosen for them indicate, adjunction and symjunction operate on conceptual roles in ways somewhat analogous to the way union and intersection operate on sets.

We can now define an interpretation of a language as a function that assigns each sentence a conceptual content.

**Definition 70** (Interpretation Function, $[\![\cdot]\!]$). An interpretation function $[\![\cdot]\!]$ maps sentences of a language $\mathfrak{L}$ to some conceptual contents $\mathbb{C}$. If $A \in \mathfrak{L}$ is an atomic sentence, then $[\![A]\!] =_{df.} \langle \mathtt{a}^+, \mathtt{a}^- \rangle \in \mathbb{C}$ . The logical-connective clauses that an interpretation must respect are as follows:

$[\![\neg A]\!] =_{df.} \langle \mathtt{a}^-, \mathtt{a}^+ \rangle$,

$[\![A \to B]\!] =_{df.} \langle \mathtt{a}^- \sqcap \mathtt{b}^+ \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^+), \mathtt{a}^+ \sqcup \mathtt{b}^- \rangle$,

$[\![A \wedge B]\!] =_{df.} \langle \mathtt{a}^+ \sqcup \mathtt{b}^+, \mathtt{a}^- \sqcap \mathtt{b}^- \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^-) \rangle$,

$[\![A \vee B]\!] =_{df.} \langle \mathtt{a}^+ \sqcap \mathtt{b}^+ \sqcap (\mathtt{a}^+ \sqcup \mathtt{b}^+), \mathtt{a}^- \sqcup \mathtt{b}^- \rangle$.

Interpretations of sets of sentences are the set of the interpretants of the sentences, that is, $[\![\Gamma]\!] = \{[\![\gamma]\!] \mid \gamma \in \Gamma\}$.

In general, these semantic clauses assign contents to logically complex sentences by assigning ranges of subjunctive robustness to their use as premises and as conclusions. The clauses place logically complex sentences in an equivalence class of sets of implications with the same range of subjunctive robustness. That makes sense because, according to implication-space semantics, what it is for a sentence to have a content is for it to be a member of an equivalence class with respect to ranges of subjunctive robustness.

To understand how interpretation functions work, it may be helpful to notice that negation swaps the premisory and conclusory roles, so that the content that an interpretation function assigns to a negation $\neg\phi$ is always the content that is like the content that the function assigns to $\phi$ except that the premisory and the conclusory roles are swapped. Hence, the ranges of subjunctive robustness of $\langle \{\neg\phi\}, \emptyset \rangle$ and $\langle \emptyset, \{\neg\phi\} \rangle$ are, respectively, the ranges of subjunctive robustness of $\langle \emptyset, \{\phi\} \rangle$ and $\langle \{\phi\}, \emptyset \rangle$. So, whatever yields a good implication when combined with $\langle \{\neg\phi\}, \emptyset \rangle$ also yields a good implication when combined with $\langle \emptyset, \{\phi\} \rangle$. This is how negation allows us to make explicit incompatibility. For we encode the fact that a set of bearers $\Gamma$ is incompatible with the bearer $\phi$ in our implications as $\langle \Gamma \cup \{\phi\}, \emptyset \rangle \in \mathbb{I}$. And our negation clauses ensure that we can substitute the negation for the negatum on the other side *salva consequentia* to get: $\langle \Gamma, \{\neg\phi\} \rangle \in \mathbb{I}$. Thus, the negation makes explicit, on the right side of an implication, that it is ruled out by what is on the left side. This is precisely the explicative potential that we described as the essential function of negation in Chapter Three.

For the interpretation of conditionals, the content of a conditional always has as its conclusory role the adjunction of the premisory role of the antecedent and the conclusory role of the consequent. That means that the range of subjunctive robustness of $\langle \Gamma, \Delta \cup \{\phi \to \psi\} \rangle$ is always identical to the range of subjunctive robustness of $\langle \Gamma \cup \{\phi\}, \Delta \cup \{\psi\} \rangle$. This ensures that the conditional makes implications explicit in the way described in Chapter Three. Namely, it ensures that $\langle \Gamma \cup \{\phi\}, \Delta \cup \{\psi\} \rangle \in \mathbb{I}$ just in case $\langle \Gamma, \Delta \cup \{\phi \to \psi\} \rangle \in \mathbb{I}$, which is a formulation of the Deduction-Detachment Condition on Conditionals from Chapter Three.

And the premisory role that an interpretation function assigns to a conditional is the symjunction of the conclusory role of the antecedent, the premisory role of the consequent, and the adjunction of these roles. That means that the range of subjunctive robustness of $\langle \Gamma \cup \{\phi \to \psi\}, \Delta \rangle$ is always the intersection of the range of subjunctive robustness of $\langle \Gamma, \Delta \cup \{\phi\} \rangle$ and the range of subjunctive robustness of $\langle \Gamma \cup \{\psi\}, \Delta \rangle$ and the range of subjunctive robustness of $\langle \Gamma \cup \{\psi\}, \Delta \cup \{\phi\} \rangle$. In this way, the semantic clauses for the conditional assign a conceptual content to conditionals by placing the use of the conditional as a premise and as a conclusion in equivalence classes with respect to ranges of subjunctive robustness, that is, by assigning the conditional its premisory and conclusory implicational role. The clauses for conjunction and disjunction can be understood in an analogous way.

We can now define models of implication-space semantics. In general, a model is a space of semantic interpretants together with an interpretation that assigns an interpretant from that space to every sentence of a given language. This suggests the following definition of models.

**Definition 71** (Models). A model, $\mathcal{M}$, is a pair $\left\langle \mathbb{C}, [\![\cdot]\!]^{\mathcal{M}} \right\rangle$ consisting of a set of contents, $\mathbb{C}$, and an interpretation function $[\![\cdot]\!]^{\mathcal{M}}$ that maps all sentences of a given language to contents in $\mathbb{C}$.

We haven't used the idea of truth anywhere in our construction in this chapter, and we also have no use for the traditional notion of truth in a model. Our basic semantic notion is not truth but implication, that is, reason relations. Accordingly, we can define a notion of implication among sets of sentences in a model.

**Definition 72** (Implication in a model, $\overset{\mathcal{M}}{\mid\!\sim}$). We say that the sentences $\Gamma$ model-theoretically imply the sentences $\Delta$ in model $\mathcal{M}$, written $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$, if and only if the corresponding entailment holds among their interpretants, $\mathtt{G} = [\![\Gamma]\!]^{\mathcal{M}}$ and $\mathtt{D} = [\![\Delta]\!]^{\mathcal{M}}$, in that model, that is, if and only if $\mathtt{G} \mathrel{|\!|\!\sim} \mathtt{D}$.

We generalize this idea from a single model to arbitrary sets of models as follows:

**Definition 73** (Implication in $m$-models, $\overset{m}{\mid\!\sim}$). Given a subset, $m$, of all implication-space models, for a given language, we say that the sentences in $\Gamma$ model-theoretically imply the sentences in $\Delta$ in $m$, written $\Gamma \overset{m}{\mid\!\sim} \Delta$, if and only if $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$ for all $\mathcal{M} \in m$.

We are now ready to explore the effects of interpreting sentences by assigning contents to them. A first and easy thing to note is that if we abstract contents from an implication frame whose implication space is given by a language, then interpreting the sentences of that language in terms of the abstracted contents in the most straightforward way yields the implication relation of the implication frame itself.

**Proposition 74.** *Let $\mathbb{C}$ be the conceptual contents abstracted from the implication frame, $\langle \mathfrak{L}, \mathbb{I} \rangle$ and, hence, $\mathbb{S} = \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$, and let $\mathcal{M}$ be the pair of $\mathbb{C}$ and an interpretation function such that $[\![\phi]\!]^{\mathcal{M}} = \mathcal{R}(\phi)$ for all $\phi \in \mathfrak{L}$. Then $\langle \Gamma, \Delta \rangle \in \mathbb{I}$ if and only if $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$.*

*Proof.* Suppose that $\langle \Gamma, \Delta \rangle \in \mathbb{I}$. Then, $\langle \Gamma, \Delta \rangle \approx F$ only if $\langle \emptyset, \emptyset \rangle \in \mathtt{RSR}(F)$ and, hence, $F \subseteq \mathbb{I}$. Hence, $\bigcup(\mathcal{R}\langle \Gamma, \Delta \rangle) \subseteq \mathbb{I}$. So, $[\![\Gamma]\!]^{\mathcal{M}} \mathrel{|\!|\!\sim} [\![\Delta]\!]^{\mathcal{M}}$. For the other direction, suppose that $[\![\Gamma]\!]^{\mathcal{M}} \mathrel{|\!|\!\sim} [\![\Delta]\!]^{\mathcal{M}}$. Then, $\bigcup(\mathcal{R}\langle \Gamma, \Delta \rangle) \subseteq \mathbb{I}$. But $\langle \Gamma, \Delta \rangle \in \mathcal{R}\langle \Gamma, \Delta \rangle$. Hence, $\langle \Gamma, \Delta \rangle \in \mathbb{I}$. ■

This proposition tells us that we can always abstract conceptual contents from an implication frame and then use them to interpret the bearers of that very implication frame, in an, as it were, homophonic way, while leaving the reason relation among the bearers of implicational roles unchanged. Thus, every implication frame interprets itself. Since every vocabulary is an implication frame (namely a lexicon and a consequence relation over it), it follows that every vocabulary interprets itself. That is reassuring, but it is of course not very illuminating. As we will see in the next subsection, however, conceptual contents can also be used to interpret the theories from the previous chapters in a way that lets us focus on their isomorphic implicational roles, that is, on rational forms.

## 5.2 Interpreting NMMS and Truth-Maker Theory

In this section, we show that the pragmatic-normative theory formulated in the language of the sequent calculus from Chapter Three and the semantic-representationalist truth-maker theory from Chapter Four can be interpreted in implication-space semantics. Both theories can be understood as having the structure of an implication-space semantics, each with its own particular extrinsic understanding of the bearers and exclusion relations that explain conceptual contents. We start with the sequent calculus for the pragmatic-normative theory and then turn to truth-maker theory.

### *5.2.1 Interpreting Normative Bilateralism*

As we have seen in the previous subsection, we can start with an implication frame, $\langle \mathtt{B}, \mathbb{I} \rangle$, abstract from it its contents, $\mathbb{C}$, and then use those contents to interpret a given language, in the sense of a set of uninterpreted sentences. Now, if the language under consideration is not simply a set of uninterpreted sentences but rather a vocabulary in the sense we introduced in earlier chapters, then the reason relations over the sentences put restrictions on viable interpretations. In particular, if we start with a base vocabulary, like those of Chapter Three, then the base consequence relation puts constraints on viable interpretations of the language. Let us make this thought precise.

Recall from Chapter Three that a base, $\mathfrak{B}$, is a vocabulary that consists of just atomic sentences $\mathfrak{L}_{\mathfrak{B}}$ and an implication relation, $\mid\!\sim_{\mathfrak{B}}$, among sets of these atomic sentences. In order to find a class of models that can capture the implication relation of such a base, we must look at models that interpret the sentences of $\mathfrak{L}_{\mathfrak{B}}$ in an appropriate way. We say that such models are fit for the base at issue.

**Definition 75** (Model fitness for base). A model, $\mathcal{M} = \left\langle \mathbb{C}, [\![\cdot]\!]^{\mathcal{M}} \right\rangle$ is fit for a base $\mathfrak{B} = \left\langle \mathfrak{L}_{\mathfrak{B}}, \mid\!\sim_{\mathfrak{B}} \right\rangle$ if and only if, for all $\langle \Gamma, \Delta \rangle \in \mid\!\sim_{\mathfrak{B}}$, the model is such that $[\![\Gamma]\!]^{\mathcal{M}} \mid\mid\!\sim [\![\Delta]\!]^{\mathcal{M}}$.

If $b$ is the set of models that are fit for the base $\mathfrak{B}$, then the implications in $b$-models ($\overset{b}{\mid\!\sim}$) are the implications that hold in all models that are fit for base $\mathfrak{B}$. If we restrict ourselves to models that are fit for the base $\mathfrak{B}$, then the consequence relation defined by the sequent calculus $\mathsf{NMMS}_{\mathfrak{B}}$ and the model-theoretic implication relation coincide.

**Theorem 76.** *For any base vocabulary* $\mathfrak{B} = \langle \mathfrak{L}_{\mathfrak{B}}, \mid\!\sim_{\mathfrak{B}} \rangle$ *and sentences* $\Gamma, \Delta \subseteq \mathfrak{L}$ *in the logically extended lexicon of the base,* $\Gamma \overset{b}{\mid\!\sim} \Delta$ *if and only if* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$. *(Appendix, Theorem 109)*

This theorem immediately entails that, relative to any base vocabulary, the sequent calculus $\mathsf{NMMS}$ is sound and complete with respect to the implication-space semantics developed in this chapter. To see why this theorem holds (without going through the details of the proof in the Appendix), it is helpful to notice that, firstly, the match between atomic sequents and the implication-space models is ensured by the restrictions to models that are fit for the base at issue. Secondly, the semantic clauses for the logical constants in implication-space semantics correspond to the sequent rules of $\mathsf{NMMS}$ in a perhaps surprisingly direct way. To see this consider the rules for the conditional, as an example:

$$\frac{\Gamma \succ \Delta, A \qquad B, \Gamma \succ \Delta \qquad B, \Gamma \succ \Delta, A}{\Gamma, A \to B \succ \Delta}\,[\mathrm{L}{\to}] \qquad \frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \to B, \Delta}\,[\mathrm{R}{\to}]$$

Since these rules are invertible, as we have shown in Chapter Three, the right rule implies that $\Gamma \succ A \to B, \Delta$ is derivable if and only if $\Gamma, A \succ B, \Delta$ is derivable. This is the Deduction-Detachment Condition from Chapter Three. It corresponds in implication-space semantics to the claim that $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A \to B, \Delta$ holds in a model if and only if $\Gamma, A \overset{\mathcal{M}}{\mid\!\sim} B, \Delta$ holds in that model. Now, $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A \to B, \Delta$ holds just in case any implication that plays the role of $\langle \Gamma, \Delta \rangle$ yields a good implication when it is combined with anything that plays the conclusory role of $A \to B$. So $\{\langle \Gamma, \Delta \rangle\}$ must be in the range of subjunctive robustness of $\langle \emptyset, \{A \to B\} \rangle$, and the same must hold for any other implications with the same role as $\langle \Gamma, \Delta \rangle$. Our semantic clauses for the conditional tell us that the role of $\langle \emptyset, \{A \to B\} \rangle$ is $\mathtt{a}^+ \sqcup \mathtt{b}^-$, that is, it is the role of $\langle \{A\}, \{B\} \rangle$. But two things with the same role share their ranges of subjunctive robustness. So everything with the role $\{\langle \Gamma, \Delta \rangle\}$ is in the range of subjunctive robustness of $\langle \emptyset, \{A \to B\} \rangle$ just in case it is in the range of subjunctive robustness of $\langle \{A\}, \{B\} \rangle$. So, $\Gamma, A \overset{\mathcal{M}}{\mid\!\sim} B, \Delta$.

For the left rule, [L→], the analogous relation holds. Because the rule is invertible, it tells us that $\Gamma, A{\to}B \succ \Delta$ is derivable if and only if all of

$\Gamma \succ A, \Delta$ and $\Gamma, B \succ \Delta$ and $\Gamma, B \succ A, \Delta$ are derivable. Suppose that $\Gamma, A \to B \overset{\mathcal{M}}{\mid\!\sim} \Delta$. Then the premisory role of $A \to B$ is such that combining anything that plays that role and anything with the role of $\langle \Gamma, \Delta \rangle$ yields a good implication. Hence, $\langle \Gamma, \Delta \rangle$ (and anything that plays its role) is in the range of subjunctive robustness of $\langle \{A \to B\}, \emptyset \rangle$. The semantic clause for the conditional informs us that the role of $\langle \{A \to B\}, \emptyset \rangle$ is $\mathtt{a}^- \sqcap \mathtt{b}^+ \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^+)$. And, as we noted above, the range of subjunctive robustness of a symjunction of roles is the intersection of the ranges of subjunctive robustness of the roles whose symjunction it is. So, the range of subjunctive robustness of $\mathtt{a}^- \sqcap \mathtt{b}^+ \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^+)$ is the intersection of the ranges of subjunctive robustness of $\langle \emptyset, \{A\} \rangle$ and $\langle \{B\}, \emptyset \rangle$ and $\langle \{B\}, \{A\} \rangle$. Therefore, $\langle \Gamma, \Delta \rangle$ (and anything that plays its role) is in the ranges of subjunctive robustness of all of $\langle \emptyset, \{A\} \rangle$ and $\langle \{B\}, \emptyset \rangle$ and $\langle \{B\}, \{A\} \rangle$ (and anything that plays the same roles). And that is just another way to say that $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A, \Delta$ and $\Gamma, B \overset{\mathcal{M}}{\mid\!\sim} \Delta$ and $\Gamma, B \overset{\mathcal{M}}{\mid\!\sim} A, \Delta$ all hold.

The other sequent rules of NMMS correspond to the other semantic clauses of implication-space semantics in an analogous way (see Appendix, Proposition 108). So, the rules of NMMS are not only equivalent to the semantic clauses of truth-maker theory, as we have shown in the previous chapter, but they are also equivalent to the semantic clauses of implication-space semantics. Indeed, we can formulate this correspondence in a general way as follows. The first element in the roles defined by the semantic clauses corresponds to the left rule in the sequent calculus, and the second element corresponds to the right rule in the sequent calculus. The roles super-scripted with a "+" stem from sentences that occur on the left in a top sequent, and the roles super-scripted with a "−" stem from sentences that occur on the right in a top sequent. An adjunction indicates that the adjoined roles stem from sentences in a single top sequent. And a symjunction indicates that the symjoined roles stem from sentences that occur in different top sequents. Given that the contexts are always shared in all the sequents of any rule application, using this correspondence, the semantic clauses above uniquely determine the sequent rules of NMMS, and the other way around.

All the virtues and features of NMMS carry over to $\overset{b}{\mid\!\sim}$. In particular, the logical vocabulary of implication-space semantics makes explicit reason relations in the same sense as NMMS. The Deduction-Detachment Condition on Conditionals, the Incoherence-Incompatibility Condition on Negation, the Antecedent-Adjunction Condition on Conjunctions, and the Succedent-Summation Condition on Disjunctions all hold in every implication-space model. Moreover, $\overset{b}{\mid\!\sim}$ can codify nonmonotonic and nontransitive implication relations; thus, it allows us to codify and theorize open reason relations. Similarly, the relation between NMMS and classical logic carries over to implication-space semantics. Let us spell out the last point in a bit more detail.

We know from our discussion of NMMS that if we start with a base consequence relation that includes all and only the instances of Containment, then the logically extended consequence relation coincides with classical propositional consequence. And if the base consequence relation includes all instances of Containment, then the logically extended consequence relation includes the classical propositional consequence relation. Hence, the following is an immediate corollary of Theorem 76.

**Corollary 77.** *If the base* $\mathfrak{B} = \left\langle \mathfrak{L}_{\mathfrak{B}}, \mathrel{\mid\!\sim_{\mathfrak{B}}} \right\rangle$ *is such that, for all* $\Gamma, \Delta \subseteq \mathfrak{L}_{\mathfrak{B}}$, *we have* $\Gamma \mathrel{\mid\!\sim_{\mathfrak{B}}} \Delta$ *just in case* $\Gamma \cap \Delta \neq \emptyset$, *then* $\models_{CL} = \mathrel{\mid\!\sim^{b}}$. *And if the base is such that, for all* $\Gamma, \Delta \subseteq \mathfrak{L}_{\mathfrak{B}}$, *we have* $\Gamma \mathrel{\mid\!\sim_{\mathfrak{B}}} \Delta$ *if* $\Gamma \cap \Delta \neq \emptyset$, *then* $\models_{CL} \subseteq \mathrel{\mid\!\sim^{b}}$.

Just as in NMMS, we can say that, for bases that obey Containment, there is a narrowly logical part of the consequence relation defined by implication-space semantics, and this narrowly logical part is simply classical logic. Given any base that obeys Containment, its narrowly logical part is the part of the logically extended consequence relation that it shares with the extensions of all other bases that obey Containment. We could also formulate this connection thus: If we say that a proper conceptual content is a conceptual content, $\mathtt{p}$, such that, for all $\mathtt{q}$, $\bigcup(\mathtt{p}^+ \sqcup \mathtt{p}^- \sqcup \mathtt{q}) \subseteq \mathbb{I}$, then the implications in any implication-space model with only proper contents include all implications of classical logic. It is supraclassical.

### *5.2.2 Interpreting Truth-Maker Theory*

We now turn to truth-maker theory. We can define an implication frame for any modalized state space, $\langle S, S^\diamond, \sqsubseteq \rangle$, by letting the bearers be worldly propositions, that is, pairs of sets of states from $S$, and defining the good implications by appeal to impossible states in the way we did in the previous chapter.

**Definition 78** (Implication frame of a modalized state space). Given a modalized state space, $\langle S, S^\diamond, \sqsubseteq \rangle$, its implication frame is $\langle \mathtt{B}, \mathbb{I}_{\bar{\diamond}} \rangle$, with $\mathtt{B} = \mathcal{P}(S) \times \mathcal{P}(S)$ being the set of worldly propositions of the state space, which we denote by $p = \langle p^+, p^- \rangle$ and the like. And $\mathbb{I}_{\bar{\diamond}} \subseteq \mathcal{P}(\mathtt{B}) \times \mathcal{P}(\mathtt{B})$ is the relation such that $\langle \{p_1, ..., p_n\}, \{q_1, ..., q_m\} \rangle \in \mathbb{I}_{\bar{\diamond}}$ if and only if $s \notin S^\diamond$, for all states $s = t_1 \Cup ... \Cup t_n \Cup u_1 \Cup ... \Cup u_m$ such that $t_i \in p_i^+$, for all $1 \leq i \leq n$, and $u_j \in q_j^-$, for all $1 \leq j \leq m$.

This definition is long but all it does is to take a modalized state space and return the set of worldly propositions in that state space and the implication relation among these propositions, according to the idea of truth-maker consequence that we introduced in the previous chapter. According to this idea, a set of worldly propositions implies another just in case any fusion of a truth-maker of each proposition in the first set and a falsity-maker of each proposition in the second set is an impossible state. With $p^+$ being the set of truth-makers and $p^-$ being the set of falsity-makers of a worldly proposition $p$, this conception of truth-maker consequence yields the implication relation $\mathbb{I}_{\bar{\diamond}}$. So, the implication frame of a modalized state space is really just the set of worldly propositions in the state space of truth-maker theory and the consequence relation among them, according to truth-maker consequence.

We can now interpret a modalized state space by its implication frame. We think of the modalized state space as providing us with a "vocabulary," that is, a "lexicon" and a consequence relation over that "lexicon" (although worldly propositions are, of course, not sentences). We then let this vocabulary interpret itself, as we did in Proposition 74 above. This yields an interpretation of truth-maker theory in implication-space semantics.

**Theorem 79.** *Let $\mathcal{M}$ be the implication-space model defined by the implication frame, $\langle \mathtt{B}, \mathbb{I}_{\bar{\diamond}} \rangle$, of the modalized state space $\langle S, S^{\diamond}, \sqsubseteq \rangle$ and the interpretation function such that $[\![x]\!]^{\mathcal{M}} = \mathcal{R}(x)$ for all $x \in \mathtt{B}$. Then $P \Vdash_{\overline{PI}} C$ holds in the modalized state space if and only if $P \overset{\mathcal{M}}{\mid\!\sim} C$. (Appendix, Theorem 110)*

This theorem says that, for any modalized state space, we can interpret its worldly propositions by assigning them contents from implication frames in a way that preserves the implication relations among these worldly propositions. Notice that the theorem does not make any use of a language in the usual sense. What we interpret is a set of worldly propositions.[9]

Notice that there seems to be a certain asymmetry between implication-space semantics and truth-maker theory in Theorem 79. For one might wonder why we cannot map the implication relation among contents in truth-maker theory, $\Vdash_{\overline{PI}}$, directly to the entailment relation among contents in implication-space semantics, $\mid\!\mid\!\!\sim$, without invoking the middle man of an interpretation in $\overset{\mathcal{M}}{\mid\!\sim}$. The answer is that truth-maker theory allows for two distinct worldly propositions, $A$ and $B$, such that, for all sets of worldly propositions $P$ and $C$, we have $P, A \Vdash_{\overline{PI}} C$ just in case $P, B \Vdash_{\overline{PI}} C$ and $P \Vdash_{\overline{PI}} A, C$ just in case $P \Vdash_{\overline{PI}} B, C$. That is, there can be two distinct worldly propositions, in a truth-maker model, that play the same roles in propositional implications. In particular, these are worldly propositions whose truth-makers and falsity-makers make the same contributions, respectively, to the impossibility or possibility of states of which they are parts. The analogous situation, however, cannot occur in implication-space semantics. That is, if, for all sets of conceptual contents $\mathtt{G}$ and $\mathtt{D}$, we have $\mathtt{G}, \mathtt{p} \Vdash\!\sim \mathtt{D}$ just in case $\mathtt{G}, \mathtt{q} \Vdash\!\sim \mathtt{D}$ and $\mathtt{G} \Vdash\!\sim \mathtt{p}, \mathtt{D}$ just in case $\mathtt{G} \Vdash\!\sim \mathtt{q}, \mathtt{D}$, then $\mathtt{p} = \mathtt{q}$. After all, a conceptual content is an equivalence class with respect to roles in good implications. So contents that play the same such roles are identical. It is only at the level of content bearers that we can have distinct items that play the same role in implications. Hence, to map a relation in implication-space semantics one-to-one to a consequence relation in a truth-maker model, we must use a space of content bearers and not conceptual contents themselves. In this sense, the worldly propositions of truth-maker theory are more like sentences than the conceptual contents of implication-space semantics: there can be distinct worldly propositions with the same content, but there cannot be distinct contents with the same content.

If we use interpretation functions on both sides, in truth-maker theory and in implication-space semantics, we can avoid this asymmetry. Indeed, it follows from Theorem 79 that if there is a truth-maker model in which exactly a particular set of implications hold among the interpreted sentences, then there is an implication-space model in which exactly the same implications hold. The theorem ensures that for every truth-maker model, there is a parallel implication frame model such that the consequence relation defined by these models coincide. To make this precise, let us define explicitly what we mean by "parallel models."

**Definition 80** (Parallel Models). Let $\mathfrak{L}$ be a language, then an implication-space model $\mathcal{M}$ and a truth-maker model $\mathcal{M}'$ are parallel just in case, for all $\phi \in \mathfrak{L}$, $[\![\phi]\!]^{\mathcal{M}} = |\phi|^{\mathcal{M}'}$ and the implication frame of $\mathcal{M}$ is the implication frame of the modalized state space of $\mathcal{M}'$.[10]

Truth-maker models and implication-space models that are parallel in this sense codify the same consequence relation over the common language that they interpret.

**Proposition 81.** *If the implication-space model $\mathcal{M}$ and truth-maker model $\mathcal{M}'$ are parallel, then $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$ just in case $\Gamma \models_{TM} \Delta$ in $\mathcal{M}'$. (Appendix, Theorem 111)*

If we are given any truth-maker model, we can construct the parallel implication-space model. We simply take the implication frame of the modalized state space of the truth-maker model and add the interpretation such that $[\![\phi]\!]^{\mathcal{M}} = |\phi|^{\mathcal{M}'}$ for all $\phi \in \mathfrak{L}$. Hence, it follows from the last theorem that if there is a truth-maker model that codifies a particular consequence relation over a language, then there is an implication-space model that codifies the same consequence relation over that language.

We can also go in the other direction. If we are given an implication-space model that codifies a particular consequence relation over a language, then we can construct a truth-maker model that codifies the same consequence relation over that language.

**Proposition 82.** *If there is an implication-space model* $\mathcal{M}$ *such that* $\mathrel{\overset{\mathcal{M}}{\mid\!\sim}} = X$*, then there is a truth-maker model* $\mathcal{M}'$ *such that* $\models_{\overline{TM}} = X$*. (Appendix, Proposition 112)*

Putting the two directions together yields:

**Theorem 83.** *There is an implication-space model* $\mathcal{M}$ *such that* $\mathrel{\overset{\mathcal{M}}{\mid\!\sim}} = X$ *if and only if there is a truth-maker model* $\mathcal{M}'$ *such that* $\models_{\overline{TM}} = X$*.*

It follows immediately from this theorem that truth-maker theory and implication-space semantics are equivalent in their power to provide counterexamples to implications. Hence, if we think of model theory as a way to provide counterexamples to implications, then these two theories are equivalent as model theories. However, we have seen that implication-space semantics provides an account of conceptual contents as roles in implications, while truth-maker theory allows for worldly propositions that play the same implicational roles. In this sense, the interpretants of implication-space semantics are more abstract; they correspond to equivalence classes of the interpretants of truth-maker theory.

With all this in place, it is easy to show that if we restrict our implication-space models to models that are fit for a material base $\mathfrak{B}$, thus looking at $\mathrel{\overset{b}{\mid\!\sim}}$ as we did in the previous section, then this yields a consequence relation that coincides with the truth-maker consequence for the same material base.

**Proposition 84.** $\Gamma \mathrel{\overset{b}{\mid\!\sim}} \Delta$ *just in case* $\Gamma \models^{\mathfrak{B}}_{\overline{TM}} \Delta$*. (Appendix, Proposition 113)*

Let us take stock. We can use implication-space semantics to interpret the sequent calculus NMMS and truth-maker theory in parallel ways. Both theories provide us with a space of bearers of implicational roles. These are sentences in NMMS and they are worldly propositions in truth-maker theory.[11] Moreover, both theories provide us with an implication relation over sets of bearers; this relation is $\mid\!\sim$ in NMMS and $\mathbb{I}_{\bar{\diamond}}$ (defined by

$S^{\diamond}$) in truth-maker theory. So, we can abstract conceptual contents from both theories. We can give accounts of the logical connectives in all three theories, and they have exactly corresponding effects in their respective theories. That is, the operational rules of NMMS and the semantic clauses for the connectives in truth-maker theory are both equivalent to the semantic clauses of implication-space semantics. So, the three theories are equivalent as logical theories. However, implication-space semantics is, in one sense, the most abstract theory of the three theories. For it is an intrinsic semantic theory: it uses reason relations itself to interpret the items that stand in these reason relations. The codification of reason relations in the metavocabulary of implication-space semantics does not explain these reason relations in terms of normative exclusion relations among discursive acts or alethic exclusion relations among worldly states. Implication-space semantics does not specify what the bearers of implicational roles are or what the nature of exclusion relations between them is. We have thus arrived at an intrinsic characterization of the rational forms that we encountered in NMMS and in truth-maker theory.

## 5.3 The Monoidal Structure of Reason Relations

Let us take one more step to an even higher level of abstraction and ask what the general structure is that allows us to interpret NMMS and truth-maker theory in implication-space conceptual role semantics. Doing so is a way to ask what general class of things can fruitfully be interpreted in implication-space semantics. And given that this implication-space semantics is the intrinsic metavocabulary of reason relations, that question is equivalent to the question of what the structural features of reason relations in general are. As it turns out, mathematically speaking, the answer to this question is that reason relations in general have the structure of sets of pairs that form a commutative monoid together with a bipartition of the monoid set. In this subsection, we want to explain this answer.

A commutative monoid is a set, M, with a binary associative and commutative operation, $\bigcirc$, on M such that there is an identity element, $e$, in M, that is, an element such that for all $m \in$ M, $m \bigcirc e = m$. Now, in the case of reason relations, the elements of our monoid are pairs of sets among which reason relations hold, and we call these pairs "candidate implications." In the theories in this book, we usually think of the operation $\bigcirc$ on candidate implications as the union of the two elements of the pairs that are candidate implications, so that $\langle x,y\rangle \bigcirc \langle u,w\rangle = \langle x \cup u, y \cup w\rangle$. However, what matters for the structure of implication-space semantics is merely that the operation $\bigcirc$ that combines candidate implications is such that $\langle x,y\rangle \bigcirc \langle u,w\rangle = \langle u,w\rangle \bigcirc \langle x,y\rangle$ and $\langle x,y\rangle \bigcirc (\langle u,w\rangle \bigcirc \langle v,z\rangle) = (\langle x,y\rangle \bigcirc \langle u,w\rangle) \bigcirc \langle v,z\rangle$: commutativity and associativity. In cases in which the items that stand in implication relations are not sets but, say, multisets (which we use when dealing with structurally noncontractive reason relations), an operation other than union must be used here, such as multiset union.[12] As long as this underlying operation is associative and commutative, the emerging operation on pairs will also be associative and commutative.

Why do we need an operation that combines candidate implications, like $\bigcirc$, in order to formulate an implication-space semantics? We need it in order to define ranges of subjunctive robustness, which are in turn what defines equivalence classes of implications and, hence, implicational roles. Why is this operation associative and commutative? Because the ordering and grouping of premises and conclusions does not matter for implications.[13]

Given the associative and commutative operation on sets of pairs, the only thing that is missing to arrive at a commutative monoid is an identity element. In the case of sets as the relata of reason relations, the identity element is the pair $\langle\emptyset,\emptyset\rangle$; for clearly $\langle x\cup\emptyset, y\cup\emptyset\rangle = \langle x,y\rangle$. In implication-space semantics, the existence of an identity element is important because it ensures that the range of subjunctive robustness of a candidate implication determines whether the candidate implication is a good implication. For it is easy to see that a candidate implication is a good implication just in case the identity element is in the candidate implication's range of subjunctive robustness. In this way, we can ensure that there cannot be two equivalent implications, that is, implications with the same range of subjunctive robustness, of which one is good and the other is not.

So far, we have seen that if we have a set of implications and the premises and conclusions of these implications are combined in a way that is associative and commutative and there is something we can combine with any premises and conclusions without changing the result, then we have a space of implications that is a commutative monoid of pairs. In order to define ranges of subjunctive robustness and, hence, implicational roles, however, we also need a bipartition that distinguishes the *good* and the *bad* implications, so that we can say that the range of subjunctive robustness of a candidate implication is the set of candidate implications that yield good implications when combined with the target candidate implication.

Let us sum up. The structure of a commutative monoid of pairs with bipartition is necessary for an implication frame. For any implication frame $\langle \mathtt{B},\mathbb{I}\rangle$, the set $\mathbb{S} = \mathcal{P}(\mathtt{B})\times\mathcal{P}(\mathtt{B})$ with the operation of $\langle x,y\rangle \bigcirc \langle u,w\rangle = \langle x\cup u, y\cup w\rangle$ and the identity element $\langle\emptyset,\emptyset\rangle$ is a commutative monoid and $\mathbb{I}$ is a particular subset of the monoid set. Going in the other direction, once we have a commutative monoid of pairs with bipartition, then we can define ranges of subjunctive robustness and, hence, implicational roles. For let $\langle \mathsf{M}, \bigcirc, \mathsf{I}\rangle$ be a commutative monoid, $\langle \mathsf{M}, \bigcirc\rangle$, and a subset $\mathsf{I} \subseteq \mathsf{M}$ (which of course defines a partition on $\mathsf{M}$). Then we can say that, for $M \subseteq \mathsf{M}$, the range of subjunctive robustness is $\mathtt{RSR}(M) = \{x : \forall m \in M\, (x \bigcirc m \in \mathsf{I})\}$. Moreover, we can say that for all $N, M \subseteq \mathsf{M}$, $N \approx M$ if and only if $\mathtt{RSR}(N) = \mathtt{RSR}(M)$. And then we can define implicational roles in the structure $\langle \mathsf{M}, \bigcirc, \mathsf{I}\rangle$ as above. The definitions of interpretation functions, models, and entailment then all follow as above. We accordingly arrive at an implication-space model.

So we have a structure in which we can identify implicational roles (in the way in which we do in implication-space semantics) just in case the structure is a commutative monoid of pairs with a bipartition. It is precisely in such structures that it makes sense to speak of bearers of implicational roles. Since reason relations are relations among bearers of implicational roles, this is the general structure of reason relations.

## 5.4 Noncontractive Reason Relations

In this section, we demonstrate the flexibility of the apparatus of implication-space semantics by extending its expressive power to capture structurally open reason relations in which Contraction fails in implication-space semantics. We discuss two examples, namely the sequent calculus $\mathsf{NMMS}_{\mathfrak{B}}^{/ctr}$ and multiplicative additive linear logic.

### *5.4.1 Noncontractive Material Consequence Relations*

In Chapter Three, we formulated not only the sequent calculus $\mathsf{NMMS}_{\mathfrak{B}}$, which allows failures of Monotonicity and Cut, but also $\mathsf{NMMS}_{\mathfrak{B}}^{/ctr}$, in which Contraction can fail. We are now in a position to formulate an implication-space conceptual role semantics for the sequent calculus $\mathsf{NMMS}_{\mathfrak{B}}^{/ctr}$. We start by thinking of implication frames as pairs of bearers and relations between multi-sets of bearers, thus thinking of implication among bearers as a relation between multi-sets rather than sets. The definitions of ranges of subjunctive robustness, adjunction, symjunction, and conceptual contents are adjusted accordingly, always replacing set-theoretic notions applied to sets of bearers with their counterparts for multi-sets, which keep track of the number of occurrences of their elements.

Next, we change the semantic clauses for the logical vocabulary, thus tweaking our notions of interpretation functions and models. A noncontractive model is a pair of such conceptual contents and a noncontractive interpretation function, which is defined as follows:

**Definition 85** (Noncontractive Interpretation Function, $[\![\cdot]\!]$). An interpretation function $[\![\cdot]\!]$ is defined inductively, and maps sentences of a language

$\mathfrak{L}$ to conceptual contents. If $A \in \mathfrak{L}$ is an atomic sentence, then $[\![A]\!] =_{df.} \langle \mathtt{a}^+, \mathtt{a}^- \rangle \in \mathbb{C}$ . The connective clauses are as follows:

$[\![\neg A]\!] =_{df.} \langle \mathtt{a}^-, \mathtt{a}^+ \rangle$,
$[\![A \rightarrow B]\!] =_{df.} \langle \mathtt{a}^- \sqcap \mathtt{b}^+, \mathtt{a}^+ \sqcup \mathtt{b}^- \rangle$,
$[\![A \barwedge B]\!] =_{df.} \langle \mathtt{a}^+ \sqcup \mathtt{b}^+, \mathtt{a}^- \sqcap \mathtt{b}^- \rangle$,
$[\![A \veebar B]\!] =_{df.} \langle \mathtt{a}^+ \sqcap \mathtt{b}^+, \mathtt{a}^- \sqcup \mathtt{b}^- \rangle$.

Interpretations of sets of sentences are the set of the interpretants of the sentences, for example: $\mathtt{G} = [\![\Gamma]\!] = \{[\![\gamma_i]\!] \mid \gamma_i \in \Gamma\}$.

Furthermore, we adjust our notion of model-theoretic implication as follows:

**Definition 86** (Non-contractive implication in $m$-models, $\overset{m}{\mid\!\sim}$). Given a subset of all non-contractive implication-space models $m$, for a given language, we say that the sentences in $\Gamma$ model-theoretically imply in $m$ the sentences in $\Delta$, written $\Gamma \overset{m}{\mid\!\sim}_{/ctr} \Delta$, if and only if $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$ for all $\mathcal{M} \in m$.

The so-defined implication-space semantics is sound and complete with respect to the sequent calculus $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$ from Chapter Three.

**Theorem 87.** *Let* $\mathfrak{B} = \left\langle \mathfrak{L}_{\mathfrak{B}}, \mid\!\sim_{\mathfrak{B}} \right\rangle$ *be a base vocabulary in which contraction may fail, let b be the set of models that are fit for* $\mathfrak{B}$*, and let* $\Gamma, \Delta \subseteq \mathfrak{L}$*. Then* $\Gamma \overset{b}{\mid\!\sim}_{/ctr} \Delta$ *if and only if* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$. *(Appendix, Theorem 114)*

This theorem says that we have an implication-space semantics for the noncontractive reason relations codified by $\mathsf{NMMS}^{/ctr}$. It follows that the representation theorem from Chapter Three holds for the consequence relations formulated in noncontractive implication-space semantics, just as it does for the sequent-calculus version. So, for every set of candidate implications in the base, there is a single candidate implication in the logically extended vocabulary that holds if and only if all the candidate implications in the base hold.

It may be noted that the semantic clauses of the implication-space semantics for $\mathsf{NMMS}^{/ctr}$ are a bit simpler than those for NMMS. This difference corresponds exactly to the difference between the sequent rules with two top sequents in $\mathsf{NMMS}^{/ctr}$ and with three top sequents in NMMS. In the sequent rules of NMMS and in the semantic clauses for NMMS, the third element in the sequent rules and the semantic clauses, respectively, is needed to ensure that Contraction is admissible.

### *5.4.2 Linear Logic*

Failures of contraction are perhaps most familiar from linear logic. Hence, one might wonder whether one can recover linear logic in our implication-space semantics. Indeed, one can. We will not consider the so-called exponential connectives or quantifiers here, thus limiting ourselves to *multiplicative additive linear logic* (known as "MALL"). More specifically, we will give semantic clauses for multiplicative conjunction ($\otimes$, also known as "tensor" or "times"), multiplicative disjunction ($\parr$, also known as "par"), additive conjunction (&, also known as "with"), and additive disjunction ($\oplus$, also known as "plus").

In order to formulate MALL in implication-space semantics, we leave the noncontractive models unchanged except for the semantic clauses. In particular, we change the definition of interpretation functions as follows:

**Definition 88** (LL Interpretation Function, $[\![\cdot]\!]$). An interpretation function $[\![\cdot]\!]$ is defined inductively, and maps sentences of a language $\mathfrak{L}$ to conceptual contents. If $A \in \mathfrak{L}$ is an atomic sentence, then $[\![A]\!] =_{df.} \langle \mathtt{a}^+, \mathtt{a}^- \rangle \in \mathbb{C}$ . The connective clauses are as follows:

$[\![\neg A]\!] =_{df.} \langle \mathtt{a}^-, \mathtt{a}^+ \rangle$,

$[\![A \otimes B]\!] =_{df.} \langle \mathtt{a}^+ \sqcup \mathtt{b}^+, \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-)) \rangle$,

$[\![A \parr B]\!] =_{df.} \langle \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^+) \sqcup \mathtt{RSR}(\mathtt{b}^+)),\ \mathtt{a}^- \sqcup \mathtt{b}^- \rangle$,

$[\![A \& B]\!] =_{df.} \langle \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^+) \cup \mathtt{RSR}(\mathtt{b}^+)),\ \mathtt{a}^- \sqcap \mathtt{b}^- \rangle$,

$[\![A \oplus B]\!] =_{df.} \langle \mathtt{a}^+ \sqcap \mathtt{b}^+,\ \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \cup \mathtt{RSR}(\mathtt{b}^-)) \rangle$,

Interpretations of sets of sentences are the set of the interpretants of the sentences, for example, $\mathtt{G} = [\![\Gamma]\!] = \{[\![\gamma_i]\!] \mid \gamma_i \in \Gamma\}$.

The two conjunctions of linear logic each share one of their roles with the conjunction of $\mathsf{NMMS}^{/ctr}$. The premisory role of a multiplicative conjunction $A \otimes B$ is the premisory role of the conjunction of $\mathsf{NMMS}^{/ctr}$ (and also of NMMS), namely $\mathtt{a}^+ \sqcup \mathtt{b}^+$. The conclusory role of the additive conjunction $A\&B$ is the conclusory role of the conjunction of $\mathsf{NMMS}^{/ctr}$, namely $\mathtt{a}^- \sqcap \mathtt{b}^-$. Similarly, the two disjunctions of linear logic each share one of their roles with the disjunction of $\mathsf{NMMS}^{/ctr}$. The conclusory role of the multiplicative disjunction $A \parr B$ is the conclusory role of the disjunction of $\mathsf{NMMS}^{/ctr}$ (and also of NMMS), namely $\mathtt{a}^- \sqcup \mathtt{b}^-$. The premisory role of the additive disjunction $A \oplus B$ is the premisory role of the disjunction of $\mathsf{NMMS}^{/ctr}$, namely $\mathtt{a}^+ \sqcap \mathtt{b}^+$.

The differences between MALL and $\mathsf{NMMS}^{/ctr}$ lie in the other roles, which are all the roles whose clauses (directly) mention ranges of subjunctive robustness, namely the conclusory roles of multiplicative conjunction and additive disjunction and the premisory roles of multiplicative disjunction and additive conjunction. All the implicational roles that are new in linear logic are defined as roles of ranges of subjunctive robustness of combinations of ranges of subjunctive robustness, where this combination can happen either by adjunction or by set union. The combination happens by adjunction for the multiplicative connectives, and it happens by set union for the additive connectives.

It is noteworthy that, for each of the four linear connectives, if we allow ourselves to replace the range of subjunctive robustness of a premisory (conclusory) role with the corresponding conclusory (premisory) role and the other way around, then we can move back and forth between the premisory and the conclusory roles by taking the range of subjunctive robustness of the role with which we begin. For instance, $\mathtt{a}^+ \sqcup \mathtt{b}^+$ is the premisory role of $A \otimes B$. Replacing the premisory roles with ranges of subjunctive robustness of conclusory roles yields $\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-)$. If we now take the role of the range of subjunctive robustness of this result, we get $\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-))$, which is the conclusory role of $A \otimes B$. The same works in the other direction, for $\mathtt{RSR}(\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-)))$ is $\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-)$, from which we can move to $\mathtt{a}^+ \sqcup \mathtt{b}^+$. This works for all four connectives. In this way, we can compute the roles that are new in linear logic, relative to $\mathsf{NMMS}^{/ctr}$, in a uniform way.

An important upshot of this last observation is that if the sentences in linear logic are such that the premisory roles are the roles of ranges of subjunctive robustness of their conclusory roles and the other way around, then this is preserved by the linear connectives. Note, however, that our negation rule, which remains unchanged in linear logic, encodes the idea that negating something amounts to swapping the premisory and the conclusory role of what one negates. Hence, it now seems that there are two ways to switch between premisory and conclusory roles of something, namely, first, by taking its negation and, second, by taking the role of its range of subjunctive robustness. This is the crucial conceptual insight that allows us to move between the implication-space semantics for NMMS and linear logic. As will become clearer below, one way to understand implication-space semantics is to view it as what happens to phase-space semantics for linear logic if one allows negation and ranges of subjunctive robustness to come apart. The result of making this distinction is increased expressive power.

With the intepretations of multiplicative and additive conjunction and disjunction defined in this way, we restrict our space of models to those in which $\bigcup(\mathtt{a}^+ \sqcup \mathtt{a}^-) \subseteq \mathbb{I}$ for all roles of sentences $[\![A]\!] = \mathtt{a}$ and, hence $A$, we have $A \overset{\mathcal{M}}{\mid\!\sim} A$, and we use "$\overset{LL}{\mid\!\sim}$" for implications that hold in all of these models. The so-defined consequence relation is the consequence relation of MALL.

**Theorem 89.** *A is a linear tautology if and only if* $\mathrel{|\!\!\stackrel{\mathcal{M}}{\sim}}$ *A in all LL implication-space models. And, hence,* $\Gamma \models \Delta$ *in* MALL *if and only if* $\Gamma \mathrel{|\!\!\stackrel{LL}{\sim}} \Delta$. *(Appendix, Theorem 128)*

It might be illuminating to give some hints regarding why this theorem is true (without going through the details of the proof in the Appendix). Girard's (1987) phase-space semantics for MALL starts with a definition of a phase space, which is a commutative monoid of set $\boldsymbol{P}$, a commutative and associative operation $\bullet$ on that set, an identity element 1, and a subset $\bot_{\boldsymbol{P}} \subseteq \boldsymbol{P}$. Girard then defines the so-called dual, $G^{\bot}$, of subsets of $\boldsymbol{P}$ as follows: $G^{\bot}$ is $\{p \in \boldsymbol{P} \mid \forall q \in G\,(p \bullet q \in \bot_{\boldsymbol{P}})\}$. That is, the dual of $G$ is the set of elements of $\boldsymbol{P}$ whose combination with anything in $G$ is in $\bot_{\boldsymbol{P}}$. So, if we think of $\boldsymbol{P}$ as the set of candidate implications and $\bot_{\boldsymbol{P}}$ as the good implications, then the dual of something is its range of subjunctive robustness. Girard then restricts his attention to so-called facts, which are subsets of $\boldsymbol{P}$ such that $G = G^{\bot\bot}$. There is a mapping between such structures and implication frames.

- A phase space corresponds to an implication frame, where $\boldsymbol{P}$ corresponds to the space of implications $\mathbb{S}$, the anti-phases $\bot_{\boldsymbol{P}}$ correspond to the good implications $\mathbb{I}$ and the identity element, 1, of the phase space corresponds to $\mathsf{e} = \langle \emptyset, \emptyset \rangle$.
- The dual of subsets of $\boldsymbol{P}$ corresponds to the range of subjunctive robustness of subsets of $\mathbb{S}$.
- Restricting our attention to facts corresponds to working with roles of candidate implications rather than simply candidate implications.

We can thus map phase spaces to implication frames and the other way around. This correspondence between phase spaces and implication frames reveals that implication-space semantics is a close relative of phase-space semantics.[14] The proof of Theorem 89 relies on this correspondence, which is developed in more detail in the Appendix.

The most important difference between implication-space semantics and phase-space semantics is that, in Girard's phase-space semantics, the dual of facts perform two tasks at once, namely the task of providing a treatment of negation and the task of serving, in effect, as ranges of subjunctive robustness. By contrast, in implication-space semantics, these two tasks are played by distinct items. The two-sidedness of implicational roles in implication-space semantics takes over the first of these tasks, while ranges of subjunctive robustness perform the second task. If we restrict our attention to LL models in implication-space semantics, the difference between these two items is elided, which comes out in the fact that in the canonical LL models, which are the models such that $\bigcup(\mathtt{a}^{+} \sqcup \mathtt{a}^{-}) \subseteq \mathbb{I}$ for all roles of sentences $[\![A]\!] = \mathtt{a}$ and nothing else is in $\mathbb{I}$, we have $\mathtt{RSR}(\mathtt{a}^{+}) = \mathtt{a}^{-}$ and $\mathtt{RSR}(\mathtt{a}^{-}) = \mathtt{a}^{+}$ (Appendix: Lemma 125). Hence, in the canonical LL models $\mathtt{RSR}(\mathtt{a}) = \neg\mathtt{a}$. This does not hold in implication-space models in general, but it is built deeply into the structure of phase-space models. Thus, implication-space semantics is a generalization of Girard's phase-space semantics, namely a generalization in which ranges of subjunctive robustness and the interpretation of negation need not coincide.

This generalization allows us to include material implications from the nonlogical base consequence relation in implication-space semantics, and it allows for failures of Cut. To see this, note that if only implications whose role has the form $\mathtt{a}^{+} \sqcup \mathtt{a}^{-}$ are good implications, as is the case in the implication-space models that codify linear logic, then the only thing in the range of subjunctive robustness of occurrences of a premisory role of some bearer are occurrences of the conclusory role of the same bearer. That is only implications whose role is that of an implication of the form $\phi \mathrel{|\!\sim} \phi$. This implies that only formally good implications are included as good implications in linear logic. Materially good implications are excluded. It is of the essence of logical expressivism to reject such an expressive restriction.

Moreover, since the instances of $\phi \mathrel{|\!\sim} \phi$ are closed under Cut, it is built into the phase structures of linear logic that Cut holds. So, the phase-space semantics for linear logic cannot capture open reason relations. If one allowed implications whose roles are not of the form $\mathtt{a}^{+} \sqcup \mathtt{a}^{-}$ to be good implications, then the range of subjunctive robustness of bearers is no longer guaranteed to coincide with the role of their negations. Hence, the dual of a fact can no longer serve as the semantic interpretant of negation. Therefore, phase-space semantics can capture neither material nor nontransitive reason relations. It is accordingly doubly unfit to serve in the logical expressivist project of codifying open-structured reason relations. Unlike phase-space semantics, implication-space semantics allows us to give separate treatments of negation and ranges of subjunctive robustness. In implication-space semantics, we can codify, for instance, a premisory role such that the range of subjunctive robustness of its instances includes more than just instances of their conclusory role, which allows us to codify materially good implications.

As a final remark on the implication-space semantics for linear logic, it is worth pointing out that the semantic clauses for linear logic given above illustrate how we can capture sequent rules in which contexts of top sequents are combined and in which sentences that do not occur anywhere in a top sequent are introduced as subformulae in the bottom sequent. As an example of the first phenomenon, note that the conclusory role of $A \otimes B$

is $\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-))$, and that this corresponds to the following right-rule in a two-sided sequent calculus for linear logic:

$$\frac{\Gamma \succ A, \Delta \qquad \Theta \succ B, \Lambda}{\Gamma, \Theta \succ A \otimes B, \Delta, \Lambda} \; [\otimes\text{R}]$$

To see how this works, it is helpful to note that $\mathtt{RSR}(\mathtt{a}^-)$ is the set of contexts $\langle \Gamma, \Delta \rangle$ such that $\Gamma \succ A, \Delta$ is a derivable sequent.[15] So $\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-)$ is the set of contexts $\langle \Gamma \cup \Theta, \Delta \cup \Lambda \rangle$ such that $\Gamma \succ A, \Delta$ and $\Theta \succ B, \Lambda$ are both derivable. Therefore, $\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-))$ as a conclusory role is the role played by sentences, $C$, such that $\Gamma, \Theta \succ C, \Delta, \Lambda$ is derivable just in case $\Gamma \succ A, \Delta$ and $\Theta \succ B, \Lambda$ are both derivable.

As an example of the second phenomenon, note that the conclusory role of $A_1 \oplus B$ is $\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \cup \mathtt{RSR}(\mathtt{b}^-))$, and that this corresponds to the following right-rule in a two-sided sequent calculus for linear logic:

$$\frac{\Gamma \succ A_i, \Delta}{\Gamma \succ A_1 \oplus A_2, \Delta} \; [\oplus\text{R}], \text{i=1 or 2}$$

As before, $\mathtt{RSR}(\mathtt{a}^-)$ is the set of contexts $\langle \Gamma, \Delta \rangle$ such that $\Gamma \succ A, \Delta$ is a derivable sequent. And $\mathtt{RSR}(\mathtt{b}^-)$ is the set of contexts $\langle \Gamma, \Delta \rangle$ such that $\Gamma \succ B, \Delta$ is derivable. So $\mathtt{RSR}(\mathtt{a}^-) \cup \mathtt{RSR}(\mathtt{b}^-)$ is the set of contexts $\langle \Gamma, \Delta \rangle$ such that either $\Gamma \succ A, \Delta$ or $\Gamma \succ B, \Delta$ is derivable. Therefore, $\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \cup \mathtt{RSR}(\mathtt{b}^-))$ as a conclusory role is the role played by sentences, $C$, such that $\Gamma \succ C, \Delta$ is derivable just in case $\Gamma \succ A, \Delta$ or $\Gamma \succ B, \Delta$ is derivable.

One consequence of the last point is that, unlike what happens in all variants of NMMS, if one extends a base consequence relation by adding the connectives of linear logic, then sequents that feature linear logic connectives do not always express a unique set of sequents in the base consequence relation. Rather, they sometimes express disjunctions of sequents in the base consequence relation. That is, the sequent featuring logically complex sentences is derivable just in case one of several distinct sets of nonlogical sequents are subsets of the base consequence relation. Thus, one might know that a sequent featuring logically complex sentences holds without being in a position to know that any particular sequent is in the base consequence relation. And while one can extend nontransitive base consequence relations by adding the logical vocabulary of linear logic, the Cut rule looms large in the motivation and applications of linear logic. Finally, the nonmonotonicity for which linear logic allows is best understood as the nonmonotonicity of not including superfluous premises, not the nonmonotonicity of defeasible material inferences. NMMS is accordingly vastly to be preferred to linear logic as an expressive logic.

To sum up, we can formulate an implication-space semantics for multiplicative additive linear logic. This illustrates how the familiar additive and multiplicative connectives can be formulated in implication-space semantics. However, it is built into the phase structures of linear logic that the role of the negation of a sentence and the role of its range of subjunctive robustness are the same. This implies that the phase-space semantics for linear logic can codify neither material nor non-transitive reason relations. As we have seen above, however, implication-space semantics can codify open reason relations that include material reason relations. So implication-space semantics is more flexible and can capture more kinds of reason relations—and is in this sense expressively more powerful—than the phase-space semantics for linear logic that originally inspired it.

## 5.5 Implicational Role Inclusion

We have noted above that two bearers of roles have the same implicational role just in case they can always be replaced for one another as premise and also as conclusion *salva consequentia*, that is, without turning a good implication into a bad one. This holds because two bearers with the same range of subjunctive robustness can always be replaced for each other, *salva consequentia*, as premises and as conclusions.

Playing the same implicational role is, however, a particularly strong version of a more general family of relations of substitutability *salva consequentia*. For it may happen that, for any $X$ and $Y$, if $X, A \mid\!\sim Y$, then $X, B \mid\!\sim Y$, so that we can always replace $A$ by $B$, *salva consequentia*, as a premise. And this may be true while we cannot always replace $B$ by $A$ or replace either of them for the other as a conclusion. Indeed, since we do not assume Cut, and in particular not a context-mixing (that is, multiplicative) version of Cut, it can happen that we can replace $A$ by $B$ as a premise but we cannot replace $B$ by $A$ as a conclusion.[16] In general, there can be different cases of substitutability *salva consequentia* that fall short of sharing a full implicational role. Since contents and implicational roles more generally are defined in terms of ranges of subjunctive robustness, we can think of such inclusions among ranges of subjunctive robustness as a kind of inclusion relation among contents and other implicational roles (although not in an immediate set-theoretic sense). In this section, we want to investigate phenomena like that of an implicational role including another implicational role. In this way, we study relations of substitutability *salva consequentia* in more detail and in greater generality. It turns out that this allows us to connect implication-space semantics to several familiar logical theories.

### *5.5.1 Foundations of Role Inclusion*

In order to study the phenomenon of substitutability *salva consequentia* of bearers of roles, it proves useful to define a notion of role inclusion.[17] The idea is that the role $\mathcal{R}(\alpha)$ is included in the role $\mathcal{R}(\beta)$ if and only if we can always replace, *salva consequentia,* something that plays the first role with something that plays the second role. This is the case if and only if the range of subjunctive robustness of anything in $\mathcal{R}(\alpha)$ is a subset of the range of subjunctive robustness of anything in $\mathcal{R}(\beta)$. To see this, take the premisory role of a bearer $A$: $\mathcal{R}(\langle\{A\},\emptyset\rangle)$. Its range of subjunctive robustness consists of all sets of candidate implications such that for all their elements, $\langle X, Y\rangle$, we have $\langle X\cup\{A\}, Y\rangle \in \mathbb{I}$. Now, suppose that all of these sets of candidate implications are also in the range of subjunctive robustness of the premisory role of some other bearer $B$. That is, they are in the range of subjunctive robustness of $\mathcal{R}(\langle\{B\},\emptyset\rangle)$. Then for every candidate implication, $\langle X, Y\rangle$, if $\langle X\cup\{A\}, Y\rangle \in \mathbb{I}$, then $\langle X\cup\{B\}, Y\rangle \in \mathbb{I}$, since $\langle X\cup\{A\}, Y\rangle \in \mathbb{I}$ holds just in case$\{\langle X, Y\rangle\}$ is in the range of subjunctive robustness of $\mathcal{R}(\langle\{A\},\emptyset\rangle)$ and analogously for $\langle X\cup\{B\}, Y\rangle \in \mathbb{I}$. In other words, for any $X$ and $Y$, if $X, A \mathrel{|\!\sim} Y$, then $X, B \mathrel{|\!\sim} Y$. Thus, we can always replace, *salva consequentia, A* as a premise with $B$ as a premise. In such a case we say that the premisory role of $B$ is included in the premisory role of $A$. In general, we say that a role is included in another role just in case the range of subjunctive robustness of the first role is a subset of the range of subjunctive robustness of the second role, in all models under consideration.

It will prove useful to generalize this idea to cases in which several roles are included in several other roles. The following way of doing this proves fruitful: Take for example the premisory role of $A$ and the conclusory role of $B$. They are included in the combined conclusory roles of $C$ and $D$, in a class of models, if and only if we have $\langle X, Y\cup\{C, D\}\rangle \in \mathbb{I}$ whenever we have $\langle X\cup\{A\}, Y\rangle \in \mathbb{I}$ and $\langle X, Y\cup\{B\}\rangle \in \mathbb{I}$. To put it differently, this role inclusion holds just in case everything that is in the range of subjunctive robustness of all the included roles is also in the range of subjunctive robustness of the adjunction of the including roles. We thus take the symjunction of the included roles and the adjunction of the including roles. This particular way for handling sets of implicational roles makes the relations below easiest to see, and it can helpfully be related to proofs in sequent calculi. The following definition makes our notion of implicational role inclusion precise:

**Definition 90** (Implicational role inclusion, $\preceq$)**.** Given a set of implication-space models, $m$, we say that the set of implicational roles $\{\mathcal{R}_1, ..., \mathcal{R}_n\}$ is included in the set of implicational roles $\{\mathcal{R}_k, ..., \mathcal{R}_l\}$ in $m$, written

$\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_l$ in $m$, just in case $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i) \subseteq \mathtt{RSR}(\bigsqcup_{j=k}^{l} \mathcal{R}_j)$ in every model in $m$.

Thus, everything that makes the symjunction of the included roles good also makes the adjunction of the including roles good. To see what this amounts to, let us start by noting some general features of implicational role inclusion. In particular, we can note how structural principles governing implications of the underlying implication space show up in role inclusions among contents. The following role inclusions hold, in a model, if and only if the implications among the bearers with these contents obey Containment, Monotonicity, and Cut, respectively[18]:

| | |
|---|---|
| Containment: | $\forall \mathtt{p} \in \mathbb{C}\ (\star \preceq \mathtt{p}^+, \mathtt{p}^-)$. |
| Monotonicity: | $\forall \mathtt{p}, \mathtt{q} \in \mathbb{C}\ (\mathtt{p}^{+/-} \preceq \mathtt{p}^{+/-}, \mathtt{q}^{+/-})$ |
| Cut: | $\forall \mathtt{p} \in \mathbb{C}\ (\mathtt{p}^+, \mathtt{p}^- \preceq \mathtt{e})$ |

To see why these are formulations of the respective structural principles, recall that $\mathtt{e} = \langle \emptyset, \emptyset \rangle$ is the empty candidate implication, and $\star = \langle \mathtt{B}, \mathtt{B} \rangle$ is the implication whose premise-set and conclusion-set is the set of all bearers of the implication frame. So the range of subjunctive robustness of $\mathtt{e}$ is the set of good implications $\mathbb{I}$. And the range of subjunctive robustness of $\star$ is the entire implication space $\mathbb{S}$. After all, $\langle \mathtt{B}, \mathtt{B} \rangle \in \mathbb{I}$ and $\langle \mathtt{B}, \mathtt{B} \rangle$ does not change by taking the set-union of its premises or conclusions with any set of bearers. So our formulation of Containment says that whenever we combine the premisory and the conclusory role of any content of a bearer in one implication (by taking their adjunction), the result has the maximal range of subjunctive robustness. That is, all implications in which any bearers with the same content occur on the left side and also on the right side are indefeasible implications. Similarly, our formulation of Monotonicity says that anything that makes good all implications in some (premisory or conclusory) role $\mathtt{p}^{+/-}$ also makes good all implications, $\mathtt{p}^{+/-} \sqcup \mathtt{q}^{+/-}$, with additional premises and conclusions. Finally, our formulation of Cut says that if an implication is made good by adjoining the premisory role, $\mathtt{p}^+$, of a content and it is also made good by adjoining the conclusory role, $\mathtt{p}^-$, of that content, then the implication has a range of subjunctive robustness that is included in the range of subjunctive robustness of $\mathtt{e}$. But since $\mathtt{RSR}(\mathtt{e}) = \mathbb{I}$, this just means that the implication is already good by itself.

We can now see that, in effect, all of our semantic clauses are statements of role identities, if we extend the clauses to cover “logically complex” roles that might not have bearers. They translate immediately into mutual role inclusions, namely as follows, where "$\preceq\succeq$" stands for a role inclusion relation that goes in both directions:

**Fact 91.** *By our semantic clauses for the implication-space semantics of* NMMS *(Definition 70),*[19] *for any contents* $\mathtt{a} = \langle \mathtt{a}^+, \mathtt{a}^- \rangle$ *and* $\mathtt{b} = \langle \mathtt{b}^+, \mathtt{b}^- \rangle$, *we have* $\mathtt{a}^+ \preceq\succeq (\neg\mathtt{a})^-$, *and* $\mathtt{a}^- \preceq\succeq (\neg\mathtt{a})^+$, *and* $\mathtt{a}^+ \sqcup \mathtt{b}^+ \preceq\succeq (\mathtt{a} \wedge \mathtt{b})^+$, *and* $\mathtt{a}^- \sqcap \mathtt{b}^- \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^-) \preceq\succeq (\mathtt{a} \wedge \mathtt{b})^-$, *and similarly for the other semantic clauses.*

For example, our semantic clauses say $[\![\neg A]\!] =_{df.} \langle \mathtt{a}^-, \mathtt{a}^+ \rangle$, so that $(\neg\mathtt{a})^- = \mathtt{a}^+$ and $(\neg\mathtt{a})^+ = \mathtt{a}^-$. The resulting mutual role inclusions are $\mathtt{a}^+ \preceq\succeq (\neg\mathtt{a})^-$ and $\mathtt{a}^- \preceq\succeq (\neg\mathtt{a})^+$. And the other semantic clauses imply mutual role inclusions in an analogous way. It may also be helpful to note the following immediate consequences of the definition of implicational role inclusion:

**Fact 92.** *For any roles* $\mathcal{R}\langle\Gamma,\Delta\rangle$ *and* $\mathcal{R}\langle\Theta,\Lambda\rangle$*:*

*(i)* $\mathcal{R}\langle\Gamma,\Delta\rangle \preceq \mathtt{e}$ *just in case* $\mathtt{RSR}\langle\Gamma,\Delta\rangle \subseteq \mathbb{I}$.

*(ii)* $\star \preceq \mathcal{R}\langle\Gamma,\Delta\rangle$ *just in case, for all implications* $\langle X, Y\rangle$, $\langle\Gamma \cup X, \Delta \cup Y\rangle \in \mathbb{I}$, *that is,* $\langle\Gamma,\Delta\rangle \in \mathbb{I}$ *holds persistently.*

*(iii)* $\mathcal{R}\langle\Gamma,\Delta\rangle \preceq \mathtt{e}, \mathcal{R}\langle\Theta,\Lambda\rangle$ *just in case* $\mathcal{R}\langle\Gamma,\Delta\rangle \preceq \mathcal{R}\langle\Theta,\Lambda\rangle$.

*(iv)* $\star, \mathcal{R}\langle\Gamma,\Delta\rangle \preceq \mathcal{R}\langle\Theta,\Lambda\rangle$ *just in case* $\mathcal{R}\langle\Gamma,\Delta\rangle \preceq \mathcal{R}\langle\Theta,\Lambda\rangle$.

Finally, we can use the notion of role inclusion just introduced to define a notion of content inclusion. Let us start with an intuitive case. There seems to be a sense in which the content of "Fido is a dog" includes the content of "Fido is a mammal." One way to spell out what this means is the following: Everything that follows from "Fido is a mammal" (together with any further premises) also follows from "Fido is a dog" (together with those same further premises). And, conversely, everything that implies "Fido is a dog" also implies "Fido is a mammal." We can formulate this idea in terms of substitution *salva consequentia* as follows: We can always substitute *salva consequentia* "Fido is a mammal" by "Fido is a dog" as premises. And we can always substitute *salva consequentia* "Fido is a dog" by "Fido is a mammal" as conclusions. And putting this in terms of role inclusion, we can say that the premisory role of "Fido is a mammal" is included in the premisory role of "Fido is a dog," and the conclusory role of "Fido is a dog" is included in the conclusory role of "Fido is a mammal." Let us formulate this idea as a general definition:

**Definition 93** (Content inclusion, $\Subset$). For any contents $\mathtt{a} = \langle \mathtt{a}^+, \mathtt{a}^- \rangle \in \mathbb{C}$ and $\mathtt{b} = \langle \mathtt{b}^+, \mathtt{b}^- \rangle \in \mathbb{C}$, we say that content $\mathtt{a}$ includes content $\mathtt{b}$, written $\mathtt{b} \Subset \mathtt{a}$, if and only if $\mathtt{b}^+ \preceq \mathtt{a}^+$ and $\mathtt{a}^- \preceq \mathtt{b}^-$.

With this notion of content inclusion in hand, we can say that two contents overlap just in case there is some third content that is included in both contents.[20]

In the remainder of this section, we show that many familiar logics can be understood as logics of implicational role inclusion. We can thus see these logics as allowing us to codify different aspects of inclusions among implicational roles and contents.

### *5.5.2 Logics Based on the Strong Kleene Scheme*

In this subsection, we show how familiar logics that use the strong Kleene truth-tables emerge as implicational role inclusions within implication-space semantics (of NMMS). These logics are Priest's paraconsistent logic of paradox (LP), the paracomplete strong Kleene logic (K3), the nontransitive strict-tolerant logic (ST), and the nonreflexive tolerant-strict logic (TS).

The logics LP, K3, ST, and TS are usually formulated by using three truth-values, often denoted by 1 (for truth), 0 (for falsehood), and $\frac{1}{2}$ for the third truth-value. All four logics are based on the strong Kleene truth-tables, which are the following (treating the conditional as defined): the value of a negation $v(\neg\phi)$ is $1 - v(\phi)$; the value of a conjunction is the minimum of the values among the conjuncts, and the value of a disjunction is the maximum of the values among the disjuncts. An interpretation is a function that assigns to each sentence of a language exactly one of these three truth-values, in a way that respects the strong Kleene truth-tables.

The only difference between LP, K3, ST, and TS is the way in which consequence is defined in them. If we define consequence as preservation of values more than 0, this yields the logic LP (Priest, 2006). And if we define consequence as preservation of value 1, then this yields the logic K3. A consequence relation holds in ST just in case every interpretation in which all the premises have value 1 is such that some conclusion has a value other than 0 (Ripley, 2012; Cobreros et al., 2012, 2020a). And a TS consequence holds just in case every interpretation in which all the premises have values other than 0 is such that some conclusion has value 1 (French, 2016; Cobreros et al., 2020a).

Famously, Priest advocates LP as a response to paradoxes, while Kripke's solution is based on K3.[21] Advocates of LP think of the third truth-value as representing truth-value-gluts, that is, as cases in which a sentence is both true and false (in a model). And advocates of K3 think of the third truth-value as representing truth-value-gaps, that is, as cases in which a sentence is neither true nor false (in a model). If we include a paradoxical sentence, $\lambda$, that receives the third truth-value in all models, then LP invalidates the principle *ex contradictione quodlibet*, so that $\lambda \wedge \neg\lambda \not\models_{LP}$ . And K3 invalidates the principle of the excluded middle, so that $\not\models_{K3} \lambda \vee \neg\lambda$.

Unless we add paradoxical sentences or the like to our language, the consequence relation of ST coincides with classical logic, and the consequence relation of TS is empty. However, one can add a transparent truth-predicate to all of these logics without making their consequence relations trivial. If one adds a paradoxical sentence, then Cut fails in ST. One can think of TS as rejecting Reflexivity (French, 2016).

It is well known that the logics LP and K3 coincide, each in its own way, with the so-called "local metainferential" consequence relation of ST (Cobreros et al., 2020a, b; Barrio et al., 2015; Dicher and Paoli, 2019). Metainferential validity is a relation between sets of candidate implications. A metainference is locally valid just in case every model that is a counterexample to all the conclusion candidate implications is a counterexample to at least one premise candidate implication. We can recover the relation of local metainferential validity in implication-space semantics. This works because there is an ST-counterexample-preserving bijection between strong Kleene valuations and a particular subset of implication-space models, which we call the conic implication-space models.

**Definition 94** (Conic implication-space models)**.** A conic implication-space model, $\mathcal{M}$, is a reflexive implication space model in which there is a pair $\langle \Gamma, \Delta \rangle$ such that $X \not\mid\!\sim^{\mathcal{M}} Y$ if and only if $\langle X, Y \rangle \in \mathcal{P}(\Gamma) \times \mathcal{P}(\Delta)$.

In a conic implication-space model, for some $\langle \Gamma, \Delta \rangle$, every bad candidate implication is such that its premise-set is a subset of $\Gamma$ and its conclusion-set is a subset of $\Delta$. Thus, the bad candidate implications form, as it were, a cone with $\langle \Gamma, \Delta \rangle$ as a top element and all other bad candidate implications sitting in a set-theoretic inclusion lattice below that top element (where the lattice structure is such that both sets of a pair must be subsets of the elements of the pairs above it). We can map such models to strong Kleene valuations, $v$, in an ST-counterexample-preserving bijection as follows: $v(A) = 1$ just in case $A \in \Gamma$, $v(A) = 0$ just in case $A \in \Delta$, and $v(A) = \frac{1}{2}$ otherwise. Intuitively, the bad implications are those with true premises and false conclusions, and $\langle \Gamma, \Delta \rangle$, at the top of the cone, consists of the pair of *all* the true premises and *all* the false conclusions. So every bad implication has premises and conclusions that are subsets of these and so are found lower in the cone on the set-inclusion lattice. It is easy to see that this bijection maps implications for which the strong Kleene model is an ST counterexample to implications that are bad in the conic implication-space model, and the other way around. For every conic implication-space model, we can construct the strong Kleene model to which it is mapped by our bijection, and the other way around. And since the consequence relation of ST is reflexive, the consequence relations of these conic implication-space models also obey Reflexivity (see Appendix, Theorem 135).

The well-known connections between ST, LP, K3, and TS emerge in implication-space semantics in terms of implicational role inclusions over conic models (see Appendix, Fact 137). In particular, LP agrees with role inclusions among conclusory roles, and K3 agrees with the converse of role inclusions among premisory roles, over all conic implication-space models. ST coincides with the role of implications that include the role of the whole implication space. And TS coincides with the inclusion of the symjunction of the roles of the conjunction of the premises and the disjunction of the conclusions in the role of the identity element.

**Proposition 95.** *Suppose that the only constraint on implication-space models is that they be conic, and let* $[\![\Gamma]\!] = \mathtt{G}$ *and* $\mathtt{G}^+ = \{\mathtt{g}^+ \mid \mathtt{g} \in \mathtt{G}\}$ *and* $\mathtt{G}^- = \{\mathtt{g}^- \mid \mathtt{g} \in \mathtt{G}\}$ *and analogously for* $[\![\Delta]\!] = \mathtt{D}$. *Then:*

$\Gamma \models_{LP} \Delta$ *if and only if* $(\bigwedge \mathtt{G})^- \preceq \mathtt{D}^-$.

$\Gamma \models_{K3} \Delta$ *if and only if* $(\bigvee \mathtt{D})^+ \preceq \mathtt{G}^+$.

$\Gamma \models_{ST} \Delta$ *if and only if* $\star \preceq \mathtt{G}^+, \mathtt{D}^-$.

$\Gamma \models_{TS} \Delta$ *if and only if* $(\bigwedge \mathtt{G})^-, (\bigvee \mathtt{D})^+ \preceq \mathtt{e}$. *(Appendix, Proposition 139)*

This means that LP is the logic of conclusory role inclusions. Thus, LP tells us which conclusions we can always replace *salva consequentia* for which other conclusions, in all conic implication-space models. Similarly, K3 tells us by which premises we can always replace *salva consequentia* which other premises, in all conic implication-space models. Since $\star \preceq \mathtt{G}^+, \mathtt{D}^-$ just in case $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$ holds persistently in all conic models, ST tells us which implications are persistently good in all conic models. Moreover, $(\bigwedge \mathtt{G})^-, (\bigvee \mathtt{D})^+ \preceq \mathtt{e}$ holds just in case $\mathtt{RSR}((\bigwedge \mathtt{G})^- \sqcap (\bigvee \mathtt{D})^+) \subseteq \mathbb{I}$. So, a consequence $\Gamma \models_{TS} \Delta$ holds just in case, for all $g \subseteq \Gamma$, if $X \overset{\mathcal{M}}{\mid\!\sim} g, Y$, then $X \overset{\mathcal{M}}{\mid\!\sim} Y$, and for all $d \subseteq \Delta$, if $X, d \overset{\mathcal{M}}{\mid\!\sim} Y$, then $X \overset{\mathcal{M}}{\mid\!\sim} Y$. Thus, TS captures which sentences can be dropped from an implication *salva consequentia*.

Since the logics LP and K3 have received considerable attention in the literature, it may be worth commenting a bit more on how they emerge in implication-space semantics. Let us start by pointing out that, from the perspective of implication-space semantics, LP and K3 are limiting cases that look only at formal implications: implications that hold in all conic models. For a candidate implication has a counterexample in a conic model if and only if it has a counterexample in the set of models that obey Containment. Thus, we lose all information about material implication and incompatibility. If we looked at fewer models, for instance, by requiring that certain material implication and incompatibility relations hold in the implications spaces of these models, we would get more substantive role inclusion relations. Thus, LP and K3 provide the lower bounds of conclusory and premisory role inclusions, respectively. It follows from this that, given Containment, the intersection of LP and K3, that is, the set of implications that hold in both logics, is a lower bound on content inclusion.

**Proposition 96.** *The content inclusion* $\mathtt{b} \Subset \mathtt{a}$ *holds in all conic models, if and only if, for all A such that* $[\![A]\!] = \mathtt{a}$ *and for all B such that* $[\![B]\!] = \mathtt{b}$, *we have* $A \models_{K3} B$ *and* $A \models_{LP} B$.

Let us now turn to the paraconsistent and paracomplete natures of LP and K3, respectively. The principles that LP and K3 reject, but that hold in classical logic, are related in intimate ways to substitution *salva consequentia*. In classical logic, we have $(\mathtt{a} \wedge \neg\mathtt{a})^- \preceq \mathtt{e}$ and $(\mathtt{a} \vee \neg\mathtt{a})^+ \preceq \mathtt{e}$. The first of these role inclusions can be viewed as a version of *ex contradictione quodlibet,* and it fails in LP. In particular, the role inclusion $(\mathtt{a} \wedge \neg\mathtt{a})^- \preceq \mathtt{e}$ says that, in all conic models, for all $\Gamma$ and all $\Delta$, if $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A \wedge \neg A, \Delta$, then $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$. The second can be viewed as a version of excluded middle, and it fails in K3. In particular, the role inclusion $(\mathtt{a} \wedge \neg\mathtt{a})^+ \preceq \mathtt{e}$ says that, in all conic models, for all $\Gamma$ and all $\Delta$, if $\Gamma, A \vee \neg A \overset{\mathcal{M}}{\mid\!\sim} \Delta$, then $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$.

Given Containment, these two role inclusions are both versions of Cut in the underlying implication space. For, $(\mathtt{a} \wedge \neg\mathtt{a})^- \preceq \mathtt{e}$ if and only if $\mathtt{a}^- \sqcap \mathtt{a}^+ \sqcap (\mathtt{a}^- \sqcup \mathtt{a}^+) \preceq \mathtt{e}$, which holds just in case $\mathtt{a}^-, \mathtt{a}^+ \preceq \mathtt{e}$ and $\mathtt{a}^- \sqcup \mathtt{a}^+ \preceq \mathtt{e}$ both hold. And $\mathtt{a}^- \sqcup \mathtt{a}^+ \preceq \mathtt{e}$ is guaranteed by Containment, which holds in all conic models. Hence, $\mathtt{a}^-, \mathtt{a}^+ \preceq \mathtt{e}$ if and only if $(\mathtt{a} \wedge \neg\mathtt{a})^- \preceq \mathtt{e}$. And as we have seen above, $\mathtt{a}^-, \mathtt{a}^+ \preceq \mathtt{e}$ means that we can always apply Cut to bearers of the content $\mathtt{a}$. Similarly, supposing Containment, Cut holds in the underlying implication relation just in case the role inclusion $(\mathtt{a} \vee \neg\mathtt{a})^+ \preceq \mathtt{e}$ holds. For, $(\mathtt{a} \vee \neg\mathtt{a})^+ \preceq \mathtt{e}$ if and only if $\mathtt{a}^+ \sqcap \mathtt{a}^- \sqcap (\mathtt{a}^+ \sqcup \mathtt{a}^-) \preceq \mathtt{e}$. By the previous reasoning, it follows that $(\mathtt{a} \vee \neg\mathtt{a})^+ \preceq \mathtt{e}$ means that we can always apply Cut to bearers of the content $\mathtt{a}$.

It might seem surprising that what looks like a truth-value glut in LP shows up as a failure of Cut in implication-space semantics, and similarly for what looks like a truth-value gap in K3. Why is that? The technical answer is this: Given our connectives and Containment, the roles $(\mathtt{a} \vee \neg\mathtt{a})^+$ and $(\mathtt{a} \wedge \neg\mathtt{a})^-$ and $\mathtt{a}^- \sqcap \mathtt{a}^+$ are all equivalent. This means that, with $[\![A]\!] = \mathtt{a}$, we can always substitute, *salva consequentia*, $A \vee \neg A$ as a premise in an implication for $A \wedge \neg A$ as a conclusion, and the other way around. And we can replace either one of them, *salva consequentia*, by two implications, which are otherwise the same, with $A$ being a premise in one implication and $A$ being a conclusion in the other implication. So, $\Gamma, A \vee \neg A \overset{\mathcal{M}}{\mid\!\sim} \Delta$ just in case $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A \wedge \neg A, \Delta$ just in case $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A, \Delta$ and $\Gamma, A \overset{\mathcal{M}}{\mid\!\sim} \Delta$ both hold. Hence, we can drop $A \vee \neg A$ as a premise or, equivalently, $A \wedge \neg A$ as a conclusion in an implication, *salva consequentia*, just in case we can apply Cut to two implications that are like the one with which we started except that $A$ figures as a premise in one and as a conclusion in the other.

When we interpret this situation in terms of normative bilateralism, we can say that there is a sentence, $A$, that can neither be accepted nor rejected, in a given position, so that Cut fails, just in case, in that position, one can neither reject $A \wedge \neg A$ nor accept $A \vee \neg A$. But if we cannot reject $A \wedge \neg A$, then it can seem that we must accept $A$ and also accept $\neg A$. And this seems to commit us to saying that $A$ is true and also false. And if we cannot accept $A \vee \neg A$, then it seems that we must reject $A$ and also reject $\neg A$. And this seems to commit us to saying that $A$ is neither true nor false.

When we interpret the situation in terms of truth-maker theory, we can say that there is a sentence, $A$, such that neither one of its truth-makers nor one of its falsity-makers can be fused with a given state into a state that is possible, just in case neither a falsity-maker of $A \wedge \neg A$ nor a truth-maker of $A \vee \neg A$ can be fused with the given state into a state that is possible. But if $A \wedge \neg A$ cannot be made false, then it can seem that neither $A$ nor $\neg A$ can be made false, so that it seems that $A$ and $\neg A$ must both be made true. And this seems to commit us to saying that $A$ is true and also false. And if $A \vee \neg A$ cannot be made true, then it seems that neither $A$ nor $\neg A$ can be made true. And this seems to commit us to saying that $A$ is neither true nor false.

Returning to the high level of abstraction of this chapter, the general interpretation is that if $(\mathtt{a} \wedge \neg\mathtt{a})^-$ is ruled out, then $\mathtt{a}^-$ and $(\neg\mathtt{a})^-$ are both ruled out. Thus, $\mathtt{a}^+$ and $(\neg\mathtt{a})^+$ seem forced upon us, which yields the impression of a truth-value glut. And if $(\mathtt{a} \vee \neg\mathtt{a})^+$ is ruled out, then $\mathtt{a}^+$ and $(\neg\mathtt{a})^+$ are both ruled out. And this yields the impression of a truth-value gap. Given the role equivalences above, we know that these situations arise just in case the role $\mathtt{a}^- \sqcap \mathtt{a}^+$ is ruled out, which shows up as a failure of Cut.

To sum up, the impression that there are truth-value gluts or truth-value gaps can be understood as arising from failures of Cut, when we think about implicational role inclusions as relations that preserve some designated truth-value. From the perspective of implication-space semantics, however, this impression of truth-value gluts or gaps is merely a reflection of the fact that LP codifies the relation of conclusory role inclusion in all conic models, and K3 codifies the converse relation of premisory role inclusion in all conic models.

A comment on our recovery of ST and TS may also be in order. We can see from the proposition above that ST and TS are not dual logics, like LP and K3, which result from each other by flipping the sides of the role inclusions and switching premisory to conclusory roles, and the other way around, for the whole sets of premises and conclusions combined by adjunction. However, ST and TS would be dual logics if the premisory role of $\mathsf{e}$ was the conclusory role of $\star$ and the other way around. Now, the premisory role of $\mathsf{e}$ is not the conclusory role of $\star$. Rather, $\mathsf{e}$ and $\star$ are the minimal and the maximal element of the set-theoretic inclusion lattice of subsets of $\mathbb{S}$.[22] This suggests that we can think of the relation between ST and TS in terms of the inversion of the ordering of a lattice. However, we will not develop this idea here.

For our purposes, it suffices to note that, in implication-space semantics, the consequence relations of ST and TS emerge as relations of implicational role inclusion in conic models. The general fact that underlies this connection, as well as the connection between implication-space semantics and LP and K3, is that there is an ST-counterexample-preserving bijection between strong Kleene interpretations and conic implication-space models.

### *5.5.3 Logics of Content and Metaphysics*

In this subsection, we want to add a final point to our demonstration of the power of implication-space semantics to recover and connect extant logical theories. There is a family of logics that have been considered as logics of content or logics of factual equivalence (Fine, 2016; Correia, 2016).[23] We now show how some of these logics are related to identity and inclusions of contents considered as implicational roles.

We begin by looking at a logic that Correia (2016) suggested as a "logic of factual equivalence." Correia is interested in determining when two sentences, $A$ and $B$, describe the same facts or states in virtue of their (propositional) logical form, written $A \approx B$. Correia offers a Hilbert-style axiomatic system for such statements, and we use "Correia's logic" for this logic. Correia's is a proper fragment of Angell's (1989; 1977) logic of analytic entailment, which Fine (2016) advocates as a logic of content. Correia's logic differs from Angell's logic in that it doesn't validate the distributive principle according to which $A \vee (B \wedge C)$ is equivalent to $(A \vee B) \wedge (A \vee C)$. However, Correia's logic does include distribution of conjunction over disjunction as an axiom (using Correia's labels).

A10 $\quad A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$

We get Angell's first-degree system for analytic equivalence by adding the distribution principle that Correia rejects as another axiom, namely:

A11 $\quad A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$

If we add A11 but not A10, we get what Correia calls his "dual" logic. We write $\vdash_c A \approx B$ to say that $A \approx B$ is a theorem of Correia's logic, and $\vdash_{dc} A \approx B$ to say that $A \approx B$ is a theorem of Correia's dual logic. We can recover both Correia's logic and his dual logic in terms of implicational role inclusions, indeed in terms of role identities, thereby giving a precise sense to their claims to be logics of content.

**Proposition 97.** *If the only constraint on models is that they obey the semantic clauses for the implication-space semantics of* NMMS *(Definition 70), then Correia's logic and his dual system are related to implicational roles as follows. For all* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*:*
$\vdash_c A \approx B$ *if and only if, in all models,* $\mathtt{a}^+ \preceq\succeq \mathtt{b}^+$, *and so* $\mathtt{a}^+ = \mathtt{b}^+$.
$\vdash_{dc} A \approx B$ *if and only if, in all models,* $\mathtt{a}^- \preceq\succeq \mathtt{b}^-$, *and so* $\mathtt{a}^- = \mathtt{b}^-$.
*(Appendix, Proposition 143)*

We can also express this in terms of content identity.

**Proposition 98.** *The contents* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$ *are identical, and so* $\mathtt{a} \ni\in \mathtt{b}$, *in all implication-space models, if and only if* $\vdash_c A \approx B$ *and* $\vdash_{dc} A \approx B$.

So while mutual implication in LP and also K3 ensures that two bearers have the same content in all conic models, a more demanding connection is needed when we include nonconic models. And this more demanding connection is that the two bearers must be equivalent in Correia's logic and Correia's dual logic. In other words, we get Correia's logic from K3 as a logic of premise substitution *salva consequentia* if, firstly, we require that substitutions must be possible in both directions and, secondly, we disallow substitutions that might not be *salva consequentia* in nonconic models. And we get Correia's dual logic from LP in the same way.

Without entering into any debate on these issues, we can note that role inclusions can be used to recover some notions that are currently popular in so-called "analytic metaphysics." Correia and Skiles (2019) offer, for instance, accounts of gounding, essence, and generalized identity on the basis of Correia's logic. When we take the versions of their notions that apply at the sentential level, we can translate them into implicational role inclusions or identities. According to Correia and Skiles a generalized identity claim of the form "For it to be the case that $A$ is for it to be the case that $B$" holds in virtue of logical form just in case $\vdash_c A \approx B$. Hence, this notion coincides with the identity of premisory roles of the bearers $A$ and $B$. Regarding essence, Correia and Skiles's notion of full factual essence turns out to coincide also with identity of premisory implicational roles, as they hold that a generalized identity claim connecting two sentences holds just in case the facts stated by those sentences are each other's full essence. It turns out that a fact is part of the essence of another fact, in their terms, just in case some adjunction of the premisory role of the sentence that states the first fact with another role is identical to the premisory role of a sentence that states the second fact.

**Fact 99.** *Factual Essence: Its being the case that A is what it is for it to be the case that B in full, in virtue of truth-functional logical form (Correia and Skiles, 2019, 651), if and only if, for* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*, we have* $\mathtt{a}^+ = \mathtt{b}^+$ *in all implication-space models. And its being the case that A is in part what it is for it to be the case that B, in virtue of truth-functional logical form (Correia and Skiles, 2019, 651), if and only if, for* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*, there is some* $\mathtt{c}^+$ *such that* $\mathtt{a}^+ \sqcup \mathtt{c}^+ = \mathtt{b}^+$ *in all implication-space models. (Appendix, Fact 144)*

If we wanted to include more than just those factual essence claims that hold in virtue of logical form, we could restrict ourselves to a subset of all implication-space models. Requiring models to respect the corresponding identities of premisory roles will then ensure that the desired claims about essences are true in all these models. We can translate the notion of grounding in a similar way.

**Fact 100.** *Strict Full Grounding: Its being the case that* $A_1, ..., A_n$ *makes it the case that B, in virtue of truth-functional logical form (Correia and Skiles, 2019, 655), if and only if, in all implication-space models, for* $[\![A_i]\!] = \mathtt{a}_i$ *and* $[\![B]\!] = \mathtt{b}$*, (i) for some* $[\![C]\!] = \mathtt{c}$*, we have* $(\mathtt{a}_1^+ \sqcup ... \sqcup \mathtt{a}_n^+) \sqcap \mathtt{c}^+ \sqcap (\mathtt{a}_1^+ \sqcup ... \sqcup \mathtt{a}_n^+ \sqcup \mathtt{c}^+) \preceq\succeq \mathtt{b}^+$ *and (ii)* $\forall 1 \leq i \leq n$ *there are no* $[\![D]\!] = \mathtt{d}$ *and* $[\![E]\!] = \mathtt{e}$ *such that* $(\mathtt{b}^+ \sqcup \mathtt{d}^+) \sqcap \mathtt{e}^+ \sqcap (\mathtt{b}^+ \sqcup \mathtt{d}^+ \sqcup \mathtt{e}^+) \preceq\succeq \mathtt{a}_i^+$*. (Appendix, Fact 145)*

Recently, Elgin (2021) has complained that Correia and Skiles's notion of generalized identity, in effect, only considers truth-makers and does not take falsity-makers into account. Elgin suggests that a generalized identity statement connecting two sentences is true just in case the two sentences have the same truth-makers and also the same falsity-makers. Elgin offers an axiomatic system for generalized identity statements that hold merely in virtue of logical form.[24] It is easy to see that Elgin's notion can be translated into implication-space semantics as follows:

**Fact 101.** *A sentence of the form "For it to be the case that A is for it to be the case that B" is true in virtue of logical form, in the sense of Elgin (2021, sec 4), if and only if, for* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*, we have* $\mathtt{a} = \mathtt{b}$*, and so* $\mathtt{a} \ni\!\in \mathtt{b}$*, in all implication space models.*

So we can see that, from the perspective of implication-space semantics, Elgin's notion of generalized identity coincides with sameness of content. It coincides with substitutability *salva consequentia* as premises and as conclusions in all models, and so merely in virtue of our semantic clauses for the logical vocabulary.

Our aim here is not to do justice to the debates in metaphysics that are related to the notions that we just recovered in implication-space semantics. Rather, we want to point out that these notions are isomorphic to notions that we can explain in terms of inclusion relations among implicational roles and, hence, ultimately in terms of substitution *salva consequentia* of bearers in implications, that is, in reason relations. In other words, we can introduce isomorphic notions by appealing just to substitutability *salva consequentia* in reason relations.

### *5.5.4 Nonlogical Role Inclusions*

We have so far used implicational-role inclusions only to capture logical relations and concepts, namely familiar logics and (analogues of) concepts from analytic metaphysics. We want to end this chapter by considering an example of the possibility of codifying *non*logical concepts in implication-space semantics. In particular, we want to show how (a variant of) a recent inferentialist theory of predicates, due to Kai Tanter (2021), can be captured by implicational-role inclusions. For this subsection, we accordingly briefly lift the general restriction of our considerations to just the sentential level, and look at subsentential expressions. However, we will not consider quantifiers, and simply include schematic letters for singular terms to talk about open sentences.

Tanter (2021) offers an account of three very general kinds of conceptual connections, and he shows how to codify conceptual connections of these kinds in a sequent calculus. The three kinds of conceptual connections are, in our terminology:

1. Conceptual equivalence: $F$ and $G$ are such that, in virtue of the meanings of $F$ and $G$, something is $F$ if and only if it is $G$.
2. Species-genus relation: $F_1, \ldots, F_n$ and $G$ are such that, in virtue of their meanings, being $F_j$ is incompatible with being $F_k$ for all $k \neq j$, and being any $F_j$ is sufficient for being $G$.

3. Property-cluster relation: $F$ and $G_1, \ldots, G_m$ are such that, in virtue of their meanings, being $F$ is sufficient to be $G_i$ for every $i$, and if something has all the properties $G_1, \ldots, G_m$, then it is $F$.

These three kinds of conceptual relations seem indeed pretty fundamental and powerful. These are the kinds of relation that allow us to build a Porphyrian tree, and do so with quite a bit of sophistication. We can, for example, codify the fact that, in virtue of the meanings of the terms, something is a computer program just in case it is a computer application. Using the second relation, we can codify that, in virtue of the meanings of the terms (let us assume), bacteria, archaea, and eukaryota are the most general species—namely the domains—of the genus of living beings. And the third kind of relations allows us to codify, in virtue of the meanings of the terms (let us assume), that something is a mammal just in case it is an animal and has hair. Inferential relations of these sorts are ubiquitous in ordinary language (colors, shapes, places...) and have accordingly loomed large in the history of thinking about the structure of concepts and reality, at least from Porphyry's *Isagoge* up through Linnaean taxonomy.

Tanter (2021) gives sequent rules for each of these three cases. We can formulate adjusted and slightly changed versions[25] of these rules, letting $\nu$ be schematic for singular terms and using double-lines to say that the sequent rule applies in both directions, so that one can not only infer the bottom sequent from the top sequents but also any top sequent from the bottom sequent. For conceptual equivalence the sequent rules are:

$$\frac{\Gamma, F\nu \succ \Delta}{\Gamma, G\nu \succ \Delta}\ \text{G/F-L} \qquad \frac{\Gamma \succ F\nu, \Delta}{\Gamma \succ G\nu, \Delta}\ \text{G/F-R}$$

This obviously ensures that $F$ and $G$ have the same premisory and conclusory implicational roles. So, we can formulate this in terms of implicational role inclusion as follows, using typewriter font for roles and superscripts for marking premisory and conclusory roles (with plus and minus, respectively):

G/F-L: $\mathtt{F}\nu^{+} \preceq\succeq \mathtt{G}\nu^{+}$

G/F-R: $\mathtt{F}\nu^{-} \preceq\succeq \mathtt{G}\nu^{-}$

This says that, for all instances of these schematic sentences, the premisory roles of $F\nu$ and $G\nu$ include each other. And the same holds for their conclusory roles. Since mutual inclusion of implicational roles implies their identity, we could also express this as: $\mathtt{F}\nu^{+} = \mathtt{G}\nu^{+}$ and $\mathtt{F}\nu^{-} = \mathtt{G}\nu^{-}$. If we used single-line sequent rules, then the inclusion relations might hold in only one direction.

The rules for the Species-Genus relation are as follows, where $F_i$ and $F_j$ stand for arbitrary but distinct $F$s such that the sequence $F_1, \ldots, F_i$ is like $F_1, \ldots, F_n$ but without $F_j$.

$$\frac{\Gamma \succ F_1\nu, \ldots, F_n\nu, \Delta}{\Gamma \succ G\nu, \Delta}\ \text{G/FR} \qquad \frac{\Gamma, F_1\nu \succ \Delta \quad \cdots \quad \Gamma, F_n\nu \succ \Delta}{\Gamma, G\nu \succ \Delta}\ \text{G/FL}$$

$$\frac{\Gamma \succ G\nu, \Delta \quad \Gamma, F_1\nu \succ \Delta \quad \cdots \quad \Gamma, F_i\nu \succ \Delta}{\Gamma \succ F_j\nu, \Delta}\ \text{RFj} \qquad \frac{\Gamma, G\nu \succ F_1\nu, \ldots, F_i\nu, \Delta}{\Gamma, F_j\nu \succ \Delta}\ \text{LFj}$$

G/FR says that it follows that something falls under one or another of the species just in case it follows that it falls under the genus. G/FL says that something follows from an object falling under any species of a genus just in case it follows from the object falling under that genus. Note that G/FR and G/FL are what you would expect if $G\nu$ is the disjunction of all the $F_k\nu$. We can express these rules in implicational role inclusion terms as follows:

G/FR: $\mathtt{F}_1\nu^- \sqcup \ldots \sqcup \mathtt{F}_\mathtt{n}\nu^- \preceq\succeq \mathtt{G}\nu^-$

G/FL: $\mathtt{F}_1\nu^+ \sqcap \ldots \sqcap \mathtt{F}_\mathtt{n}\nu^+ \preceq\succeq \mathtt{G}\nu^+$

The rule LFj says that all the species are mutually exclusive: It follows from something falling under the genus, given premises $\Gamma$, that either it falls under one of the species $F_1, \ldots, F_i$ or one of the sentences $\Delta$ holds if and only if it follows from it falling under the remaining species $F_j$, given premises $\Gamma$, that one of the sentences $\Delta$ holds. Notice that RFj and LFj are what you would expect if $F_j$ is equivalent to $G\nu \wedge \neg F_1\nu \wedge \ldots \wedge \neg F_i\nu$. We can formulate these rules in role inclusion terms as follows:

RFj: $\mathtt{G}\nu^- \sqcap \mathtt{F}_1\nu^+ \sqcap \ldots \sqcap \mathtt{F}_\mathtt{i}\nu^+ \preceq\succeq \mathtt{F}_\mathtt{j}\nu^-$

LFj: $\mathtt{G}\nu^+ \sqcup \mathtt{F}_1\nu^- \sqcup \ldots \sqcup \mathtt{F}_\mathtt{i}\nu^- \preceq\succeq \mathtt{F}_\mathtt{j}\nu^+$

The rules for the Property-Cluster relation are the following:

$$\frac{\Gamma \succ G_1\nu, \Delta \quad \cdots \quad \Gamma \succ G_m\nu, \Delta}{\Gamma \succ F\nu, \Delta}\ \text{F/GR} \qquad \frac{\Gamma, G_1\nu, \ldots, G_m\nu \succ \Delta}{\Gamma, F\nu \succ \Delta}\ \text{F/GL}$$

These are the rules one would expect if one wanted to say that $F\nu$ has the same inferential role as $G_1\nu \wedge \ldots \wedge G_m\nu$. For F/GL corresponds to the usual conjunction left-rule and its inverse, and F/GR corresponds to the usual conjunction right-rule and its inverse. We can formulate these rules in role inclusion terms as follows:

F/GR: $\mathtt{G}_1\nu^- \sqcap ... \sqcap \mathtt{G}_\mathtt{m}\nu^- \preceq\succeq \mathtt{F}\nu^-$

F/GL: $\mathtt{G}_1\nu^+ \sqcup ... \sqcup \mathtt{G}_\mathtt{m}\nu^+ \preceq\succeq \mathtt{F}\nu^+$

One can read the role inclusions as holding *de jure*, and then they say, for example, that if you want to have concepts of genus and species, then you need to define these concepts as given by the appropriate role inclusions. But we can also read the role inclusions, as it were, *a posteriori* or *de facto*, and then they say, for example, that if you find concepts whose roles are related in the given ways, then you have found yourself a genus and some species concepts. The formalism is, of course, neutral on which of these readings is more appropriate (if either).

Like the sequent rules and the implication-role inclusions for logical connectives, we can interpret the sequent rules for these conceptual connections in truth-maker theory, and doing so might add a potentially illuminating perspective. Interpreted in truth-maker theory, the rule G/F-R says that all instantiations for $\nu$ of $F\nu$ and $G\nu$ have the same falsity-makers. G/F-L says that, for every instantiation of $\nu$, $F\nu$ and $G\nu$ have the same truth-makers. So given these rules, $F\nu$ and $G\nu$ always represent the same worldly proposition.

Turning to the rules for species-genus relations, the rule G/FR says that, for every instantiation of $\nu$, a falsity-maker of $G\nu$ is a fusion of falsity-makers of each of $F_1\nu, ..., F_n\nu$. We could express this by saying that to fail to fall under a genus is to fail to fall under any of its species. The rule G/FL says that, for every instantiation of $\nu$, the truth-makers of $G\nu$ are the union of truth-makers for any of $F_1\nu, ..., F_n\nu$. We could express this by saying that to fall under a genus is to fall under one of its species. The RFj rule says that, for every instantiation of $\nu$, a falsity-maker of $F_j\nu$ is either a falsity maker of $G\nu$ or a truth-maker of one of the $F_1\nu, ..., F_i\nu$. We could express this by saying that to fail to fall under a particular species is either to fail to fall under the genus or to belong to one of the other species under that genus. The LFj rule says that, for every instantiation of $\nu$, a truth-maker of $F_j\nu$ is a fusion of a truth-maker of $G\nu$ and falsity-makers for each of the $F_1\nu, ..., F_i\nu$. We could express that by saying that to fall under a particular species is to fall under the genus and fail to fall under any of its other species.

Finally, regarding the rules for property-cluster relations, the rule F/GR says that, for every instantiation of $\nu$, a falsity-maker of $F\nu$ is anything that is a falsity-maker for one of $G_1\nu, ..., G_m\nu$. We could express this by saying that to fail to fall under a property-cluster concept is for something to fail to have one (or more) of the properties in the cluster that defined the property-cluster concept. And the rule F/GL says that, for every instantiation of $v$, a truth-maker of $Fv$ is a fusion of truth-makers for each of the $G_1v, \ldots, G_mv$. We could express that by saying that to fall under a property-cluster concept is to have all the properties in the cluster.

To sum up, the metalanguage of implicational-role inclusions allows us to formulate what reason relations must be like for some concepts in these reason relations to stand in the relations of conceptual equivalence, the species-genus relation, or the property-cluster relation. The correspondence between implication-space semantics, sequent calculi, and truth-maker theory applies to these cases in the same way in which it applies to logical vocabulary. This illustrates how implication-space semantics can codify the kind of chain of being familiar since Porphyry's *Isagoge*, at least until Linnaean taxonomy, in an abstract metalanguage of reason relations that captures the structure that is shared between more concrete characterizations of reason relations in (the right kind of) pragmatic-normative and semantic-representationalist metavocabularies.

## 5.6 Conclusion

In this chapter, we explicated reason relations in a metavocabulary of implicational roles. This explication of reason relations is intrinsic because it does not appeal to anything that is not already given in the reason relations themselves. In particular, it does not appeal to norms governing discursive acts or modal relations among worldly states. Rather, implication-space semantics takes implication relations among some particular items and abstracts from them the implicational role of these items. The pragmatic-normative account of reason relations in terms of the sequent calculus NMMS from Chapter Three and the semantic-representationalist account in terms of truth-makers from Chapter Four can both be interpreted in implication-space semantics. Indeed, we can use the same implicational roles to interpret the role of a sentence in the pragmatic-normative account and to interpret the worldly proposition that the sentence represents in the semantic-representationalist account. We have thus arrived at an abstract theory of the rational forms that are shared by sentences and the worldly propositions that they represent: a theory that abstracts away from the different ways in which the rational form can be realized in the matter of discursive acts or the matter of worldly states. We have arrived at a pure theory of rational forms.

Let us rehearse the main points in the development of implication-space semantics. We start with a space of bearers of implicational roles and an implication relation between sets of these bearers. We call this an implication frame. We then consider under which additions a candidate implication would turn into a good implication, or remain good (if it already is a good implication). We call the set of these additions the "range of subjunctive robustness" of the candidate implication. The implicational role of an implication is then the class of (sets of) implications with the same range of subjunctive robustness. The premisory role of a bearer is the role of the implication in which the bearer occurs only as a premise. The conclusory role of a bearer is the role of the implication in which the bearer occurs only as a conclusion. And the implicational role of a bearer is the pair of its premisory and its conclusory roles.

Any pair that could be the implicational role of a bearer is a conceptual content. And we can interpret languages by assigning to each sentence of the language such a conceptual content. A model is a set of contents that interpret the sentences of a given language. And a model-theoretic implication holds, in a model, just in case all implications in the role of the implication are good implications (in the implication frame of the contents of the model). Finally, a model-theoretic implication holds in a set of models if it holds in each of the models.

The mathematical structure that is necessary and sufficient to define an implication frame is a commutative monoid, in which the monoid set is a set of pairs, together with a bipartition of the monoid set. The elements of the monoid set are candidate implications, and the operation on them allows us to combine implications with each other. Ranges of subjunctive robustness correspond to the set of candidate implications that one can combine with a target implication so that the result is in the particular subset of the monoid set. Hence, the fundamental structure of reason relations is the structure of a commutative monoid in which the monoid set is a set of pairs, together with a partition of the monoid set.

We define the implicational roles of logically complex sentences in terms of the implicational roles of their constituent sentences. And we have shown that if we define the connectives in a particular way, and we look only at implication-space models in which the material implications of some base $\mathfrak{B}$ hold, then $\mathsf{NMMS}_{\mathfrak{B}}$ is sound and complete with respect to the consequence relation of implication-space semantics over the set of these models, even if the reason relations of the base are radically open-structured. If we define the connectives in other ways and allow for failures of Contraction, we can capture $\mathsf{NMMS}_{\mathfrak{B}}^{/ctr}$ and multiplicative additive linear logic in implication-space semantics.

Finally, we can define inclusion relations among implicational roles by saying that one role includes another if the range of subjunctive robustness of the first role is a subset of the range of subjunctive robustness of the second role. This notion of implicational role inclusion allows us to recover within the framework of implication-space semantics all of: the logic of paradox, strong Kleene logic, strict-tolerant logic, tolerant-strict logic, and several relations and notions from analytic metaphysics.

Thus, once we reach the level of pure rational forms, all the familiar logics and notions just mentioned fall into place as particular aspects or variations on relations among these rational forms—that is, relations among implicational roles. For instance, the ideas of truth-value gluts and gaps and the ideas of factual equivalence, essence, and grounding can all be understood in terms of implicational role inclusions. And the connectives of linear logic can also be defined in terms of implicational roles.

We have shown how one can use implication-space semantics to formulate a broad variety of familiar logics and formal theories more generally. We hypothesize, however, that there are many more such theories, and that one can extend and generalize our results in many directions. One promising project is to extend implication-space semantics to cover more logical vocabulary, such as quantifiers, standard modal operators, and other kinds of conditionals. Another is to investigate where and how the connections between implication-space semantics and the sequent calculus and truth-maker theory may break down. Yet another is to recover further familiar logics in the framework of implicational-role inclusions, with Angell's logic of analytic contaiment, the logic of first-degree entailment, and connexive logics suggesting themselves as proximal targets. These and many similar projects seem to us to be tractable and potentially fruitful research projects, which we take to support the idea that implication-space semantics can provide the tools for moving many debates forward.

Implication-space semantics is a formal inferentialist semantics. For according to implication-space semantics, the contents of sentences are the roles they play in implications, that is, the roles they play in reason relations. The semantic clauses for the logical connectives show in what sense such an inferentialist semantic theory is compositional, namely in the sense that the implicational roles of logically complex sentences are defined in terms of the implicational roles of their constituents. The semantic theory is, nevertheless, holistic because the role of an implication is defined in terms of which additions of further premises and conclusions would yield a good implication. In this way, the implicational role of an implication points beyond the implication's constituents and to other implications. Hence, the implication-space semantics of the logical connectives gives us a concrete example of a semantics that is holistic but also compositional. Moreover, implication-space semantics shows how an inferentialist semantics can capture open reason relations in a formally rigorous way, thus codifying substructural consequence relations. Finally, the many connections between implication-space semantics and extant logical and metaphysical theories suggest that implication-space semantics may serve as a unifying framework for many different approaches, from normative bilateralism, to truth-maker theory, three-valued logics, linear logic, the logics of grounding and essence and generalized identity. We take this expressive flexibility and power to be indirect evidence that implication-space semantics captures the fundamental structure of reason relations and, hence, the fundamental structure of conceptual content.

## 5.7 Appendix

### *5.7.1 Roles and Ranges of Subjunctive Robustness*

We start by listing some simple propositions about roles and ranges of subjunctive robustness, which are useful when one is trying to find one's way into implication-space semantics.

**Proposition 102.** *If* $\alpha = \mathtt{RSR}(\mathtt{RSR}(\alpha))$ *and* $\beta = \mathtt{RSR}(\mathtt{RSR}(\beta))$ *and* $\alpha$ *and* $\beta$ *both are members of some role* $\mathcal{R}$*, then* $\alpha = \beta$.

*Proof.* Clearly, if $\alpha = \mathtt{RSR}(\mathtt{RSR}(\alpha))$ and $\beta = \mathtt{RSR}(\mathtt{RSR}(\beta))$ and $\mathtt{RSR}(\alpha) = \mathtt{RSR}(\beta)$, then $\alpha = \beta$. But, by the definition of roles, if $\alpha$ and $\beta$ both are members of some role $\mathcal{R}$, then $\mathtt{RSR}(\alpha) = \mathtt{RSR}(\beta)$. ■

**Proposition 103.** $\mathtt{RSR}(\mathcal{R}(\alpha)) = \mathtt{RSR}(\alpha)$.

*Proof.* By the definition of implicational roles, $\forall x \in \mathcal{R}(\alpha)$ $(\mathtt{RSR}(\alpha) = \mathtt{RSR}(x))$. But $\mathtt{RSR}(X) = \bigcap_{x \in X} \mathtt{RSR}(x)$ and, hence, $\mathtt{RSR}(\mathcal{R}(\alpha)) = \mathtt{RSR}(x)$, for any $x \in \mathcal{R}(\alpha)$. So $\mathtt{RSR}(\alpha) = \mathtt{RSR}(\mathcal{R}(\alpha))$. ■

**Proposition 104.** $\mathcal{R}(\mathcal{R}(\alpha)) = \mathcal{R}(\alpha)$.

*Proof.* By the definition of roles, $\mathcal{R}(\mathcal{R}(\alpha))$ is the set $\{x \mid \mathtt{RSR}(\mathcal{R}(\alpha)) = \mathtt{RSR}(x)\}$. By Proposition 103, this is the set $\{x \mid \mathtt{RSR}(\alpha) = \mathtt{RSR}(x)\}$. But that is $\mathcal{R}(\alpha)$. ■

### *5.7.2 Implication-Space Semantics and NMMS*

We now move on to proving that the results of taking an adjunction or symjunction of two implicational roles is independent of the way in which these roles are represented. That is, if two items have the same implicational roles, then we do not get different results for the two items when we compute adjunctions and symjunctions of their roles.

**Proposition 105.** *If* $\mathcal{R}(F) = \mathcal{R}(F')$*, then* $\mathcal{R}(F) \sqcup \mathcal{R}(G) = \mathcal{R}(F') \sqcup \mathcal{R}(G)$.

*Proof.* Suppose that $\mathcal{R}(F) = \mathcal{R}(F')$. To show that $\mathcal{R}(F) \sqcup \mathcal{R}(G) = \mathcal{R}(F') \sqcup \mathcal{R}(G)$ it suffices to show that $\mathtt{RSR}(\{\langle f^+ \cup g^+, f^- \cup g^- \rangle \mid \langle f^+, f^- \rangle \in F, \langle g^+, g^- \rangle \in G\})$ is the same as $\mathtt{RSR}(\{\langle f'^+ \cup g^+, f'^- \cup g^- \rangle \mid \langle f'^+, f'^- \rangle \in F', \langle g^+, g^- \rangle \in G\})$. So, suppose that $\langle x, y \rangle \in \mathtt{RSR}(\mathcal{R}(F) \sqcup \mathcal{R}(G))$. Then, for every $\langle g^+, g^- \rangle \in G$, we have $\langle x \cup g^+, y \cup g^- \rangle \in \mathtt{RSR}(F)$. Since $\mathtt{RSR}(F) = \mathtt{RSR}(F')$, for every $\langle g^+, g^- \rangle \in G$, we have $\langle x \cup g^+, y \cup g^- \rangle \in \mathtt{RSR}(F')$. So $\langle x, y \rangle \in \mathtt{RSR}(\mathcal{R}(F') \sqcup \mathcal{R}(G))$. And the same reasoning works in the other direction. Therefore, $\langle x, y \rangle \in \mathtt{RSR}(\mathcal{R}(F) \sqcup \mathcal{R}(G))$ just in case $\langle x, y \rangle \in \mathtt{RSR}(\mathcal{R}(F') \sqcup \mathcal{R}(G))$. So $\mathcal{R}(F) \sqcup \mathcal{R}(G) = \mathcal{R}(F') \sqcup \mathcal{R}(G)$. ∎

**Proposition 106.** *If $\mathcal{R}(F) = \mathcal{R}(F')$, then $\mathcal{R}(F) \sqcap \mathcal{R}(G) = \mathcal{R}(F') \sqcap \mathcal{R}(G)$.*

*Proof.* If $\mathcal{R}(F) = \mathcal{R}(F')$, then $F' \approx F$. Now, $\mathcal{R}(F) \sqcap \mathcal{R}(G) = \mathcal{R}(F \cup G)$, and the latter is the set of sets of implications such that each of the sets has the same range of subjunctive robustness as $F \cup G$. So it suffices to show that, for all $\langle X, Y \rangle$, we have $\forall \langle \Gamma, \Delta \rangle \in F \cup G\ (\langle \Gamma \cup X, \Delta \cup Y \rangle \in \mathbb{I})$ iff $\forall \langle \Gamma', \Delta' \rangle \in F' \cup G\ (\langle \Gamma' \cup X, \Delta' \cup Y \rangle \in \mathbb{I})$. This holds because $F' \approx F$ entails that $\forall \langle \Gamma, \Delta \rangle \in F\ (\langle \Gamma \cup X, \Delta \cup Y \rangle \in \mathbb{I})$ iff $\forall \langle \Gamma', \Delta' \rangle \in F'\ (\langle \Gamma' \cup X, \Delta' \cup Y \rangle \in \mathbb{I})$. ∎

The following lemma about the connection between adjunction and symjunction will prove useful below.

**Lemma 107.** $\bigcup((\mathtt{a} \sqcap \mathtt{b}) \sqcup \mathtt{c}) \subseteq \mathbb{I}$ *just in case* $\bigcup(\mathtt{a} \sqcup \mathtt{c}) \subseteq \mathbb{I}$ *and* $\bigcup(\mathtt{b} \sqcup \mathtt{c}) \subseteq \mathbb{I}$.

*Proof.* It follows from the definition of symjunction that $\mathtt{RSR}(\mathtt{a} \sqcap \mathtt{b})$ is the intersection of $\mathtt{RSR}(\mathtt{a})$ and $\mathtt{RSR}(\mathtt{b})$. Hence, for all $c \in \mathtt{c}$, $c \subseteq \mathtt{RSR}(\mathtt{a} \sqcap \mathtt{b})$ if and only if $c \subseteq \mathtt{RSR}(\mathtt{a})$ and $c \subseteq \mathtt{RSR}(\mathtt{b})$. But, in general, for any role $\mathtt{z}$, we have $\bigcup(\mathtt{z} \sqcup \mathtt{c}) \subseteq \mathbb{I}$ just in case, for all $c \in \mathtt{c}$, $c \subseteq \mathtt{RSR}(\mathtt{z})$. So, $\bigcup((\mathtt{a} \sqcap \mathtt{b}) \sqcup \mathtt{c}) \subseteq \mathbb{I}$ just in case, for all $c \in \mathtt{c}$, $c \subseteq \mathtt{RSR}(\mathtt{a} \sqcap \mathtt{b})$. And the latter holds just in case, for all $c \in \mathtt{c}$, $c \subseteq \mathtt{RSR}(\mathtt{a})$ and $c \subseteq \mathtt{RSR}(\mathtt{b})$. The latter holds if and only if $\bigcup(\mathtt{a} \sqcup \mathtt{c}) \subseteq \mathbb{I}$ and $\bigcup(\mathtt{b} \sqcup \mathtt{c}) \subseteq \mathbb{I}$. ∎

We now turn to the relation between implication-space semantics and the sequent calculus NMMS from Chapter Three.

**Proposition 108.** *If there is an* NMMS *rule application with top sequents $\Gamma_1 \succ \Delta_1$, ..., $\Gamma_n \succ \Delta_n$ and bottom sequent $\Gamma_0 \succ \Delta_0$, then, for all implication–space models, $\Gamma_0 \overset{\mathcal{M}}{\mid\!\sim} \Delta_0$ holds just in case all of $\Gamma_1 \overset{\mathcal{M}}{\mid\!\sim} \Delta_1$ and ... and $\Gamma_n \overset{\mathcal{M}}{\mid\!\sim} \Delta_n$ hold.*

*Proof*. We do only the cases for negation and the conditional, as conjunction and disjunction can be defined in terms of negation and the conditional (and also the proofs work analogously for them). We take each of the rules in turn, always letting $[\![A]\!] = \langle \mathtt{a}^+, \mathtt{a}^- \rangle$ and $[\![B]\!] = \langle \mathtt{b}^+, \mathtt{b}^- \rangle$ and $[\![\Gamma]\!] = \mathtt{P} = \{\mathtt{p}_0, ..., \mathtt{p}_n\}$ and $[\![\Delta]\!] = \mathtt{C} = \{\mathtt{c}_0, ..., \mathtt{c}_m\}$.

[L¬] concludes $\Gamma, \neg A \succ \Delta$ from $\Gamma \succ A, \Delta$. By our semantic clauses for negation, $[\![\neg A]\!] = \langle (\neg \mathtt{a})^+, (\neg \mathtt{a})^- \rangle = \langle \mathtt{a}^-, \mathtt{a}^+ \rangle$. Suppose that $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A, \Delta$ and, hence, $\mathtt{P} \mid\!\mid\!\sim \mathtt{a}, \mathtt{C}$. That is, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^- \sqcup \mathtt{a}^-) \subseteq \mathbb{I}$. But $\mathtt{a}^- = (\neg \mathtt{a})^+$. So, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^- \sqcup (\neg \mathtt{a})^+) \subseteq \mathbb{I}$. Hence, $\mathtt{P}, \neg \mathtt{a} \mid\!\mid\!\sim \mathtt{C}$ and, so, $\Gamma, \neg A \overset{\mathcal{M}}{\mid\!\sim} \Delta$. And the same reasoning works in the reverse direction as well.

[R¬] concludes $\Gamma \succ \neg A, \Delta$ from $\Gamma, A \succ \Delta$. Suppose that $\Gamma, A \overset{\mathcal{M}}{\mid\!\sim} \Delta$ and, hence, $\mathtt{P}, \mathtt{a} \mid\!\mid\!\sim \mathtt{C}$. That is, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \mathtt{a}^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$. But $\mathtt{a}^+ = (\neg \mathtt{a})^-$. So, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup (\neg \mathtt{a})^- \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$. Hence, $\mathtt{P} \mid\!\mid\!\sim \neg \mathtt{a}, \mathtt{C}$ and, so, $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \neg A, \Delta$. And the same reasoning works in the reverse direction as well.

[L→] concludes $\Gamma, A \to B \succ \Delta$ from $\Gamma \succ \Delta, A$ and $\Gamma, B \succ \Delta$ and $\Gamma, B \succ \Delta, A$. By our semantic clauses for the conditional $[\![A \to B]\!] = \langle (\mathtt{a} \to \mathtt{b})^+, (\mathtt{a} \to \mathtt{b})^- \rangle = \langle \mathtt{a}^- \sqcap \mathtt{b}^+ \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^+), \mathtt{a}^+ \sqcup \mathtt{b}^- \rangle$. Suppose that $\Gamma, A \to B \overset{\mathcal{M}}{\mid\!\sim} \Delta$ and, hence, $\mathtt{P}, \mathtt{a} \to \mathtt{b} \mid\!\mid\!\sim \mathtt{C}$. That is, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup (\mathtt{a}^- \sqcap \mathtt{b}^+ \sqcap (\mathtt{a}^- \sqcup \mathtt{b}^+)) \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$. By Lemma 107, that holds just in case $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \mathtt{a}^- \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$ and $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \mathtt{b}^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$ and $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \mathtt{a}^- \sqcup \mathtt{b}^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$. Which holds just in case $\mathtt{P} \mid\!\mid\!\sim \mathtt{a}, \mathtt{C}$ and $\mathtt{P}, \mathtt{b} \mid\!\mid\!\sim \mathtt{C}$ and $\mathtt{P}, \mathtt{b} \mid\!\mid\!\sim \mathtt{a}, \mathtt{C}$. So, $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A, \Delta$ and $\Gamma, B \overset{\mathcal{M}}{\mid\!\sim} \Delta$ and $\Gamma, B \overset{\mathcal{M}}{\mid\!\sim} A, \Delta$. And the same reasoning works in the reverse direction as well.

[R→] concludes $\Gamma \succ A \to B, \Delta$ from $\Gamma, A \succ B, \Delta$. Suppose that $\Gamma, A \overset{\mathcal{M}}{\mid\!\sim} B, \Delta$ and, hence, $\mathtt{P}, \mathtt{a} \mid\!\mid\!\sim \mathtt{b}, \mathtt{C}$. That is, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup \mathtt{a}^+ \sqcup \mathtt{b}^- \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$. But $\mathtt{a}^+ \sqcup \mathtt{b}^- = (\mathtt{a} \to \mathtt{b})^-$. So, $\bigcup(\bigsqcup_{i=0}^{n} \mathtt{p}_i^+ \sqcup (\mathtt{a} \to \mathtt{b})^- \sqcup \bigsqcup_{j=0}^{m} \mathtt{c}_j^-) \subseteq \mathbb{I}$. Hence, $\mathtt{P} \mid\!\mid\!\sim \mathtt{a} \to \mathtt{b}, \mathtt{C}$ and, so, $\Gamma \overset{\mathcal{M}}{\mid\!\sim} A \to B, \Delta$. And the same reasoning works in the reverse direction as well. ■

As a matter of terminology we let *b* be the set of models that are fit for the base $\mathfrak{B}$ (see the main text for the definition of fitness of a model for a base).

**Theorem 109.** *For any base* $\mathfrak{B} = \left\langle \mathfrak{L}_{\mathfrak{B}}, \mathrel{\mid\!\!\sim_{\mathfrak{B}}} \right\rangle$ *and sentences* $\Gamma, \Delta \subseteq \mathfrak{L}$ *in the logically extended lexicon of the base,* $\Gamma \mathrel{\overset{b}{\mid\!\!\sim}} \Delta$ *if and only if* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$.

*Proof.* By Proposition 108, implications in implication-space models are closed under the rules of NMMS. And if $\Gamma \mathrel{\mid\!\!\sim_{\mathfrak{B}}} \Delta$ in the base consequence relation, then, by definition, $\Gamma \mathrel{\mid\!\!\sim} \Delta$ holds in all models fit for $\mathfrak{B}$. So if $\Gamma \succ \Delta$ is derivable in $\mathsf{NMMS}_{\mathfrak{B}}$, then $\Gamma \mathrel{\overset{b}{\mid\!\!\sim}} \Delta$. For the other direction, if $\Gamma \succ \Delta$ is not derivable in $\mathsf{NMMS}_{\mathfrak{B}}$, then a proof-search on $\Gamma \succ \Delta$ yields atomic sequents that are not in the base consequence relation $\mathrel{\mid\!\!\sim_{\mathfrak{B}}}$. We can construct an implication-space *b*-model in which these sequents fail. In fact, the model with the implication frame $\left\langle \mathfrak{L}_{\mathfrak{B}}, \mathrel{\mid\!\!\sim_{\mathfrak{B}}} \right\rangle$ and the interpretation $[\![\phi]\!]^{\mathcal{M}} = \mathcal{R}(\phi)$ for all $\phi \in \mathfrak{L}$ can serve as a counterexample. ■

### 5.7.3 *Implication-Space Semantics and Truth-Maker Theory*

We now turn to the relation between implication-space semantics and the truth-maker theory from Chapter Four.

**Theorem 110.** *Let* $\mathcal{M}$ *be the implication-space model defined by the implication frame,* $\langle \mathtt{B}, \mathbb{I}_{\bar{\diamond}} \rangle$, *of the modalized state space* $\langle S, S^{\diamond}, \sqsubseteq \rangle$ *and the interpretation function such that* $[\![x]\!]^{\mathcal{M}} = \mathcal{R}(x)$ *for all* $x \in \mathtt{B}$. *Then* $P \mathrel{\Vert\!\!\equiv_{PI}} C$ *holds in the modalized state space if and only if* $P \mathrel{\overset{\mathcal{M}}{\mid\!\!\sim}} C$.

*Proof.* Let $P = \{p_0, ..., p_n\}$ and $C = \{c_0, ..., c_m\}$, and suppose that $P \mathrel{\Vert\!\!\equiv_{PI}} C$ holds in a modalized state space, i.e., any fusion of verifiers for each $\langle p_i^+, p_i^- \rangle \in P$ and falsifiers for each $\left\langle c_j^+, c_j^- \right\rangle \in C$ is an impossible state. So, by the definition of the implication frame of a modalized state space, $\langle P, C \rangle \in \mathbb{I}_{\bar{\diamond}}$. By the reasoning from the proof of Proposition 74 above, $\langle P, C \rangle \in \mathbb{I}_{\bar{\diamond}}$ if and only if $P \mathrel{\overset{\mathcal{M}}{\mid\!\!\sim}} C$. ■

**Proposition 111.** *If the implication-space model* $\mathcal{M}$ *and truth-maker model* $\mathcal{M}'$ *are parallel, then* $\Gamma \mathrel{\overset{\mathcal{M}}{\mid\!\!\sim}} \Delta$ *just in case* $\Gamma \mathrel{\models_{TM}} \Delta$ *in* $\mathcal{M}'$.

*Proof.* Let $\mathcal{M}''$ be the model with the same implication frame, $\langle \mathtt{B}, \mathbb{I}_{\bar{\diamond}} \rangle$, as $\mathcal{M}$ but an interpretation function that interprets worldly propositions by themselves, namely the interpretation such that $[\![x]\!]^{\mathcal{M}''} = \mathcal{R}(x)$ for all $x \in \mathtt{B}$. And let $f$ be a function that takes sets of sentences of $\mathfrak{L}$ to the sets of their interpretations in $\mathcal{M}'$. We now show that implications in $\mathcal{M}$ and

$\mathcal{M}'$ both coincide with implications in $\mathcal{M}''$, thus using $\mathcal{M}''$ as our "middle man."

$\mathcal{M}$ and $\mathcal{M}''$: If $\llbracket\Gamma\rrbracket^{\mathcal{M}} = \llbracket f(\Gamma)\rrbracket^{\mathcal{M}''}$ and $\llbracket\Delta\rrbracket^{\mathcal{M}} = \llbracket f(\Delta)\rrbracket^{\mathcal{M}''}$, then $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$ just in case $f(\Gamma) \overset{\mathcal{M}''}{\mid\!\sim} f(\Delta)$. For, $\mathcal{M}$ and $\mathcal{M}''$ differ only in the language and interpretation function. Since the interpretations of $\mathcal{M}$ and $\mathcal{M}'$ are parallel, $\llbracket A\rrbracket^{\mathcal{M}} = |A|^{\mathcal{M}'} = \mathcal{R}(A)$ and so $\mathcal{R}(|A|^{\mathcal{M}'}) = \mathcal{R}(\mathcal{R}(A))$. Moreover, we defined $\mathcal{M}''$ such that $\left\llbracket |A|^{\mathcal{M}'} \right\rrbracket^{\mathcal{M}''} = \mathcal{R}(|A|^{\mathcal{M}'})$. It is easy to see, however, that $\mathcal{R}(A) = \mathcal{R}(\mathcal{R}(A))$ (Proposition 104). Hence, $\llbracket A\rrbracket^{\mathcal{M}} = \mathcal{R}(|A|^{\mathcal{M}'}) = \llbracket f(\{A\})\rrbracket^{\mathcal{M}''}$, which implies that $\llbracket\Gamma\rrbracket^{\mathcal{M}} = \llbracket f(\Gamma)\rrbracket^{\mathcal{M}''}$ and $\llbracket\Delta\rrbracket^{\mathcal{M}} = \llbracket f(\Delta)\rrbracket^{\mathcal{M}''}$. Therefore, $\Gamma \overset{\mathcal{M}}{\mid\!\sim} \Delta$ just in case $f(\Gamma) \overset{\mathcal{M}''}{\mid\!\sim} f(\Delta)$.

$\mathcal{M}'$ and $\mathcal{M}''$: To show that $\Gamma \models_{\overline{TM}} \Delta$ in $\mathcal{M}'$ just in case $f(\Gamma) \overset{\mathcal{M}''}{\mid\!\sim} f(\Delta)$, we first note that, by Theorem 79, $P \Vdash_{\overline{PI}} C$ holds in the modalized state space of all three models if and only if $P \overset{\mathcal{M}''}{\mid\!\sim} C$. Hence, it suffices to note that $\llbracket\Gamma\rrbracket^{\mathcal{M}'} = f(\Gamma)$ and $\llbracket\Delta\rrbracket^{\mathcal{M}'} = f(\Delta)$ holds by the definition of $f$. ■

**Proposition 112.** *If there is an implication-space model $\mathcal{M}$ such that $\overset{\mathcal{M}}{\mid\!\sim} = X$, then there is a truth-maker model $\mathcal{M}'$ such that $\models_{\overline{TM}} = X$.*

*Proof.* Suppose that there is an implication-space model $\mathcal{M}$ such that $\overset{\mathcal{M}}{\mid\!\sim} = X$. The modalized state space of the desired model $\mathcal{M}'$ is the space $\langle S, S^{\diamond}, \sqsubseteq\rangle$ defined as follows: Let $S$ be the set of all pairs such that $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle$ with $\Gamma^{At}$ and $\Delta^{At}$ being sets of atomic sentences of the language, $\mathfrak{L}$, that $\mathcal{M}$ interprets. Let $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle \sqsubseteq \left\langle \Theta^{At}, \Lambda^{At}\right\rangle$ if and only if $\Gamma^{At} \subseteq \Theta^{At}$ and $\Theta^{At} \subseteq \Lambda^{At}$. Let $S^{\diamond}$ be the set such that $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle \in S^{\diamond}$ if and only if $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle \notin X$. The interpretation function of $\mathcal{M}'$ is the following: For all atomic sentences $\phi$ in $\mathfrak{L}$, let $|\phi|^{+} = \{\langle\{\phi\}, \emptyset\rangle\}$ and $|\phi|^{-} = \{\langle\emptyset, \{\phi\}\rangle\}$. We extend this interpretation of atomic sentences by the usual truth-maker clauses for the logical vocabulary.

Notice that since fusion is the least upper bound with respect to parthood, the fusion of $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle$ and $\left\langle \Theta^{At}, \Lambda^{At}\right\rangle$ is $\left\langle \Gamma^{At} \cup \Theta^{At}, \Delta^{At} \cup \Lambda^{At}\right\rangle$. Hence, $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle \notin S^{\diamond}$ just in case the fusion of the truth-makers for each element in $\Gamma^{At}$ and the falsity-makers for each element in $\Delta^{At}$ is an impossible state. Hence, for atomic sentences, $\left\langle \Gamma^{At}, \Delta^{At}\right\rangle \notin S^{\diamond}$ just in case $\Gamma^{At} \models_{\overline{TM}} \Delta^{At}$.

We now consider the semantic clauses. We do just the case for negation and conjunction, as the other connectives can be defined in terms of these. We argue by induction on the number of connectives in a candidate implication.

Negation: Suppose that $\Gamma \mathrel{|\!\overset{\mathcal{M}}{\sim}} \neg A, \Delta$. By the clauses of implication-space semantics this holds if and only if $\Gamma, A \mathrel{|\!\overset{\mathcal{M}}{\sim}} \Delta$. Hence, by our induction hypothesis, $\Gamma, A \models_{TM} \Delta$. But by the semantic clauses of truth-maker theory, this holds just in case $\Gamma \models_{TM} \neg A, \Delta$. The case of $\Gamma, \neg A \mathrel{|\!\overset{\mathcal{M}}{\sim}} \Delta$ is analogous.

Conjunction: Suppose that $\Gamma \mathrel{|\!\overset{\mathcal{M}}{\sim}} A \wedge B, \Delta$. By the clauses of implication-space semantics this holds if and only if $\Gamma \mathrel{|\!\overset{\mathcal{M}}{\sim}} A, \Delta$ and $\Gamma \mathrel{|\!\overset{\mathcal{M}}{\sim}} B, \Delta$ and $\Gamma \mathrel{|\!\overset{\mathcal{M}}{\sim}} A, B, \Delta$. Hence, by our induction hypothesis, $\Gamma \models_{TM} A, \Delta$ and $\Gamma \models_{TM} B, \Delta$ and $\Gamma \models_{TM} A, B, \Delta$. But by the semantic clauses of truth-maker theory, this holds just in case $\Gamma \models_{TM} A \wedge B, \Delta$.

Suppose that $\Gamma, A \wedge B \mathrel{|\!\overset{\mathcal{M}}{\sim}} \Delta$. By the clauses of implication-space semantics this holds if and only if $\Gamma, A, B \mathrel{|\!\overset{\mathcal{M}}{\sim}} \Delta$. Hence, by our induction hypothesis, $\Gamma, A, B \models_{TM} \Delta$. But by the semantic clauses of truth-maker theory, this holds just in case $\Gamma, A \wedge B \models_{TM} \Delta$. ■

**Proposition 113.** $\Gamma \mathrel{|\!\overset{b}{\sim}} \Delta$ *just in case* $\Gamma \models^{\mathfrak{B}}_{TM} \Delta$.

*Proof.* We prove a more general result, namely that if we restrict ourselves, in both theories, to models in which a given set of implications hold, then the implications that hold in all these models, in the two theories, coincide. So, let $x \subseteq \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$, and consider only implication-space models and truth-maker models such that, for all $\langle \Gamma, \Delta \rangle \in x$, $\Gamma \mathrel{|\!\overset{x}{\sim}} \Delta$ and $\Gamma \models^{x}_{TM} \Delta$.

The consequence relations of every implication-space model and every truth-maker model are closed under the rules of NMMS and the inverted rules, which we collectively call $RUL$, because the semantic clauses of these theories are equivalent to $RUL$ (as we have seen above). And if the consequence relation of every model in a set is closed under a set of rules, then the intersection of these consequence relations is closed under these rules. Hence, the consequence relations defined by any sets of models in both theories are closed under $RUL$. Let $RUL(x)$ be the closure of a set of sequents $x$ under $RUL$. So, for all $\langle \Gamma, \Delta \rangle \in RUL(x)$, $\Gamma \mathrel{|\!\overset{x}{\sim}} \Delta$ and $\Gamma \models^{x}_{TM} \Delta$.

For the other direction, suppose that $\langle \Gamma, \Delta \rangle \notin RUL(x)$. Then we can construct counterexamples to $\Gamma \mathrel{|\!\overset{x}{\sim}} \Delta$ and $\Gamma \models^{x}_{TM} \Delta$, and we can construct these countermodels in a way that ensures that all the implications in $x$ hold in them. In fact, the model that includes all and only the implications in $RUL(x)$ is a counterexample. Therefore, $\Gamma \mathrel{|\!\overset{x}{\sim}} \Delta$ if and only if $\Gamma \models^{x}_{TM} \Delta$.

Since restricting our attention to models in which a base consequence relation holds is a special case of restricting our attention to models in which a particular set of implications hold, it follows that $\Gamma \overset{b}{\mid\!\sim} \Delta$ just in case $\Gamma \models^{\mathfrak{B}}_{TM} \Delta$. ■

### 5.7.4 Implication-Space Semantics for the Noncontractive Variant of NMMS

We now turn to the noncontractive variant of our sequent calculus, namely $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$.

**Theorem 114.** *Let $\mathfrak{B} = \left\langle \mathfrak{L}_{\mathfrak{B}}, \mid\!\sim_{\mathfrak{B}} \right\rangle$ be a base vocabulary in which contraction may fail, let b be the set of models that are fit for $\mathfrak{B}$, and let $\Gamma, \Delta \subseteq \mathfrak{L}$. Then $\Gamma \overset{b}{\mid\!\sim}_{/ctr} \Delta$ if and only if $\Gamma \succ \Delta$ is derivable in $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$.*

*Proof.* Soundness: If $\Gamma \succ \Delta$ is derivable in $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$, then $\Gamma \overset{b}{\mid\!\sim}_{/ctr} \Delta$. We argue by induction on the proof height of the derivation of $\Gamma \succ \Delta$. Since all models in $b$ are fit for $\mathfrak{B}$, the base case is immediate. For the induction step, $\Gamma \succ \Delta$ comes by one of the operational rules of $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$. Since the other connectives can be defined in terms of negation and the conditional in $\mathsf{NMMS}^{/ctr}_{\mathfrak{B}}$, it suffices to consider the rules for negation and the conditional.

Suppose $\Gamma \succ \Delta$ is $\Gamma \succ \neg A, \Delta'$ and comes by $\Gamma, A \succ \Delta'$. By our induction hypothesis, $\Gamma, A \overset{b}{\mid\!\sim}_{/ctr} \Delta'$. So for all $\mathcal{M} \in b$, labeling $\mathtt{G} = [\![\Gamma]\!]^{\mathcal{M}} = \{\mathtt{g}_0, \ldots, \mathtt{g}_n\}$ and $\mathtt{D}' = [\![\Delta']\!]^{\mathcal{M}} = \{\mathtt{d}_0, \ldots, \mathtt{d}_m\}$ and $\mathtt{a} = [\![A]\!]^{\mathcal{M}}$, we have $\mathtt{G}, \mathtt{a} \mid\!\mid\!\sim \mathtt{D}$, which means that $\bigcup(\mathtt{a}^+ \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^-) \subseteq \mathbb{I}$. Now, by our semantic clauses, $\mathtt{a}^+ = (\neg\mathtt{a})^-$. Hence, $\bigcup((\neg\mathtt{a})^- \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^-) \subseteq \mathbb{I}$ and, so, $\mathtt{G} \mid\!\mid\!\sim \neg\mathtt{a}, \mathtt{D}$. Therefore, $\Gamma \overset{b}{\mid\!\sim}_{/ctr} \neg A, \Delta'$, as desired. The case for the negation left-rule is analogous.

Suppose that $\Gamma \succ \Delta$ is $\Gamma \succ A \rightarrow B, \Delta'$ and comes by $\Gamma, A \succ B, \Delta'$. By our induction hypothesis, $\Gamma, A \overset{b}{\mid\!\sim}_{/ctr} B, \Delta'$. So for all $\mathcal{M} \in b$, labeling as before and $\mathtt{b} = [\![B]\!]^{\mathcal{M}}$, we have $\mathtt{G}, \mathtt{a} \mid\!\mid\!\sim \mathtt{b}, \mathtt{D}$, which means that $\bigcup(\mathtt{a}^+ \sqcup \mathtt{b}^- \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^-) \subseteq \mathbb{I}$. Now, by our semantic clauses, $\mathtt{a}^+ \sqcup \mathtt{b}^- = (\mathtt{a} \rightarrow \mathtt{b})^-$. Hence, $\bigcup((\mathtt{a} \rightarrow \mathtt{b})^- \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^+ \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^-) \subseteq \mathbb{I}$ and, so, $\mathtt{G} \mid\!\mid\!\sim \mathtt{a} \rightarrow \mathtt{b}, \mathtt{D}$. Therefore, $\Gamma \overset{b}{\mid\!\sim}_{/ctr} A \rightarrow B, \Delta'$, as desired.

Suppose that $\Gamma \succ \Delta$ is $\Gamma', A \rightarrow B \succ \Delta$ and comes by $\Gamma' \succ A, \Delta$ and $\Gamma', B \succ \Delta$. By our induction hypothesis, $\Gamma' \overset{b}{\mid\!\sim}_{/ctr} A, \Delta$ and $\Gamma', B \overset{b}{\mid\!\sim}_{/ctr} \Delta$. By reasoning that is analogous to the reasoning in the previous two cases, this implies that $\bigcup(\mathtt{a}^{-} \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^{+} \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^{-}) \subseteq \mathbb{I}$ and $\bigcup(\mathtt{b}^{+} \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^{+} \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^{-}) \subseteq \mathbb{I}$. By our semantic clauses, $\mathtt{a}^{-} \sqcap \mathtt{b}^{+} = (\mathtt{a} \rightarrow \mathtt{b})^{+}$. But, for any $\mathtt{x}$, we have $\bigcup(\mathtt{a}^{-} \sqcap \mathtt{b}^{+} \sqcup \mathtt{x}) \subseteq \mathbb{I}$ just in case $\bigcup(\mathtt{a}^{-} \sqcup \mathtt{x}) \subseteq \mathbb{I}$ and $\bigcup(\mathtt{b}^{+} \sqcup \mathtt{x}) \subseteq \mathbb{I}$ (by Lemma 107). So, $\mathtt{G}, \mathtt{a} \rightarrow \mathtt{b} \Vdash\!\sim \mathtt{D}$. Therefore, $\Gamma, A \rightarrow B \mathrel{\overset{b}{\mid\!\sim}}_{/ctr} \Delta'$, as desired.

Completeness: Suppose that $\Gamma \succ \Delta$ is not derivable in $\mathsf{NMMS}_{\mathfrak{B}}^{/ctr}$. Then a proof-search on $\Gamma \succ \Delta$ yields an atomic sequent that is not in $\mathfrak{B}$. We can choose a model, $\mathcal{M}' \in b$, in which these atomic sequents fail, so that if $\Theta_0 \succ \Pi_0$ is an atomic sequent that results from the proof-search but is not in $\mathfrak{B}$, then $\Theta_0 \mathrel{\overset{\mathcal{M}'}{\not\mid\!\sim}}_{/ctr} \Pi_0$. The latter implies that $\Gamma \mathrel{\overset{\mathcal{M}'}{\not\mid\!\sim}}_{/ctr} \Delta$. For, we can show that if there is a rule-application of $\mathsf{NMMS}_{\mathfrak{B}}^{/ctr}$ that concludes some sequent $\Gamma' \succ \Delta'$ and one of the top sequents of this rule-application does not hold in a model, then $\Gamma' \succ \Delta'$ does not hold in that model. For example, if $\Gamma, A \mathrel{\overset{\mathcal{M}}{\not\mid\!\sim}}_{/ctr} \Delta'$, then $\Gamma \mathrel{\overset{\mathcal{M}}{\not\mid\!\sim}}_{/ctr} \neg A, \Delta'$; for if $\bigcup(\mathtt{a}^{+} \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^{+} \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^{-}) \not\subseteq \mathbb{I}$, then $\bigcup((\neg\mathtt{a})^{-} \sqcup \bigsqcup_{i=0}^{n} \mathtt{g}_i^{+} \sqcup \bigsqcup_{j=0}^{m} \mathtt{d}_j^{-}) \not\subseteq \mathbb{I}$. And analogous reasoning applies to applications of the other rules. Therefore, $\Gamma \mathrel{\overset{b}{\not\mid\!\sim}}_{/ctr} \Delta$, as desired. ■

### *5.7.5 Implication-Space Semantics for Multiplicative Additive Linear Logic*

We now turn to multiplicative additive linear logic ($\mathsf{MALL}$), and we start with a formulation of $\mathsf{MALL}$.

**Definition 115** (Phase space). A phase space $\boldsymbol{P}$ consists of (i) a commutative monoid, which is a set $\boldsymbol{P}$ and an operation $\bullet$ defined on its elements, such that $\forall p, q, r \in \boldsymbol{P}$ ($p \bullet 1 = 1 \bullet p = p$ and $p \bullet q = q \bullet p$ and $(p \bullet q) \bullet r = p \bullet (q \bullet r)$); and (ii) a set of anti-phases $\perp_{\boldsymbol{P}}$.

**Definition 116** (Dual). If $G \subseteq \boldsymbol{P}$, then its dual $G^{\perp}$ is $\{p \in \boldsymbol{P} \mid \forall q \in G\,(p \bullet q \in \perp_{\boldsymbol{P}})\}$.

**Definition 117** (Fact). A *fact* is a subset $G$ of $\boldsymbol{P}$ such that $G^{\perp\perp} = G$; the elements of $G$ are called the phases of $G$; $G$ is valid when $1 \in G$.

**Definition 118** (Connectives). $\neg G =_{df} G^{\perp}$; $G \otimes H =_{df} (G \bullet H)^{\perp\perp}$; $G ⅋ H =_{df} (G^{\perp} \bullet H^{\perp})^{\perp}$; $G \& H =_{df} G \cap H$; $G \oplus H =_{df} (G \cup H)^{\perp\perp}$.

**Definition 119.** (i) A *phase* structure, $S$, for the propositional language with the connectives above consists in a phase space $\langle \boldsymbol{P}, \perp_{\boldsymbol{P}} \rangle$ and, for each propositional letter $a$, a fact $a_S$ of $\boldsymbol{P}$.
(ii) With each proposition we associate its interpretation, i.e., a fact in $\boldsymbol{P}$; the interpretation of $A$ in the structure $S$ is denoted by $A_S$ or $S(A)$.

(iii) $A$ is valid in $S$ if and only if $1 \in S(A)$,
(iv) $A$ is a linear tautology if and only if $A$ is valid in any phase structure $S$.

**Definition 120** (MALL Consequence ($\models_{LL}$)). $A_1, ..., A_n \models_{LL} B_1, ..., B_m$ if and only if $A_1^{\perp} \parr ... \parr A_n^{\perp} \parr B_1 \parr ... \parr B_m$ is a linear tautology.

A couple of lemmas turn out to be useful.

**Lemma 121.** *There is a mapping, $f$, between implication frames and phase spaces such that* $\mathtt{e} \in \mathtt{RSR}(G)$ *if and only if* $1 \in (f(G))^{\perp}$.

*Proof.* The following is such a mapping, letting $G \circ H =_{df} \{\langle X \cup Y, Z \cup W\rangle \mid \langle X, Z\rangle \in G\ \langle Y, W\rangle \in H\}$:

1. $f(\mathbb{S}) = \boldsymbol{P}$
2. $f(G \circ H) = f(G) \bullet f(H)$
3. $f(\mathtt{e}) = 1$
4. $f(\perp_{\boldsymbol{P}}) = \mathbb{I}_{\mathcal{M}}$

The image of an implication frame under $f$ satisfies the conditions on phase structures, as it is a commutative monoid with a set of anti-phases. Under this mapping: $f(\mathtt{RSR}(\cdot)) = (f(\cdot))^{\perp}$. For $\mathtt{RSR}(G)$ is defined as $\{x \in \mathbb{S} \mid \forall y \in G\,(x \circ y \in \mathbb{I})\}$, and the image of this set under $f$ is $\{x \in P \mid \forall y \in f(G)\,(x \bullet y \in \perp)\}$, which is the definition of $(f(G))^{\perp}$. Hence, $\mathtt{e} \in \mathtt{RSR}(G)$ iff $f(\mathtt{e}) \in f(\mathtt{RSR}(G))$ iff $1 \in (f(G))^{\perp}$. ■

**Corollary 122.** $\mathtt{e} \in \mathtt{RSR}(G \circ H)$ *iff* $1 \in (f(G) \bullet f(H))^{\perp}$.

**Lemma 123.** $\mathtt{RSR}(G \circ H) = \mathtt{RSR}(\mathtt{g} \sqcup \mathtt{h})$.

*Proof.* By the definition of adjunction, $\mathtt{g} \sqcup \mathtt{h}$, is $\mathcal{R}(G \circ H)$. The role $\mathcal{R}(G \circ H)$ is the class of things whose range of subjunctive robustness is $\mathtt{RSR}(G \circ H)$. So $x \in \mathtt{g} \sqcup \mathtt{h}$ iff $\mathtt{RSR}(G \circ H) = \mathtt{RSR}(x)$. Since the range of subjunctive robustness of a set is the intersection of the ranges of subjunctive robustness of its members, $\mathtt{RSR}(\mathtt{g} \sqcup \mathtt{h})$ is $\bigcap\{\mathtt{RSR}(x) \mid x \in \mathtt{g} \sqcup \mathtt{h}\}$, which is just $\mathtt{RSR}(G \circ H)$. Therefore, $\mathtt{RSR}(G \circ H) = \mathtt{RSR}(\mathtt{g} \sqcup \mathtt{h})$. ■

**Corollary 124.** $\mathtt{e} \in \mathtt{RSR}(\mathtt{g} \sqcup \mathtt{h})$ *if and only if* $1 \in (f(G) \bullet f(H))^{\perp}$.

**Lemma 125.** *In the canonical LL models, for all sentences $A$ such that* $[\![A]\!] = \mathtt{a}$, $\mathtt{RSR}(\mathtt{a}^{+}) = \mathtt{a}^{-}$ *and* $\mathtt{RSR}(\mathtt{a}^{-}) = \mathtt{a}^{+}$.

*Proof.* In the canonical LL models, all good implications can be represented in the form $\bigcup(\mathtt{a}^+ \sqcup \mathtt{a}^-) \subseteq \mathbb{I}$. So if $\bigcup(\mathtt{a}^+ \sqcup \mathtt{x}) \subseteq \mathbb{I}$, then $\mathtt{x} = \mathtt{a}^-$. Hence, $\mathtt{RSR}(\mathtt{a}^+) = \mathtt{a}^-$. And the analogous reasoning works for $\mathtt{RSR}(\mathtt{a}^-) = \mathtt{a}^+$. ■

**Lemma 126.** *For all phase structures S and the canonical LL models that are connected by a mapping like f above: If* $1 \in A_S$ *iff* $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^-)$ *and* $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^+)$ *iff* $1 \in A_S^\perp$ *both hold for the sentences and roles mentioned in the following statements, then these statements hold (with square brackets indicating alternatives that also hold, when uniformly applied in a given statement).*

*(a)* $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^{+[/-]} \sqcup \mathtt{b}^{+[/-]})$ *iff* $1 \in (A_S^{[\perp]} \bullet B_S^{[\perp]})^\perp$

*(b)* $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^{-[/+]} \sqcap \mathtt{b}^{-[/+]})$ *iff* $1 \in (A_S^{[\perp]} \cap B_S^{[\perp]})$

*(c)* $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^{+[/-]} \cup \mathtt{b}^{+[/-]}))$ *iff* $1 \in (A_S^{[\perp]} \cup B_S^{[\perp]})^{\perp\perp}$

*Proof.* (ad a) The first fact is immediate from Corollary 124 and Lemma 125.

(ad b) $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^- \sqcap \mathtt{b}^-)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^-) \cap \mathtt{RSR}(\mathtt{b}^-)$ iff $1 \in (A_S \cap B_S)$. And Lemma 125, ensures that $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^+ \sqcap \mathtt{b}^+)$ iff $1 \in (A_S^\perp \cap B_S^\perp)$.

(ad c) $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^+ \cup \mathtt{b}^+))$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^+) \cap \mathtt{RSR}(\mathtt{b}^+))$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^-) \cup \mathtt{RSR}(\mathtt{b}^-)$ iff $1 \in (A_S \cup B_S)$ iff $1 \in (A_S \cup B_S)^{\perp\perp}$. ■

**Proposition 127.** *In the canonical LL models and for all phase structures, letting* $f(\mathtt{a}) = A_S$ *for all roles and facts,* $1 \in A_S$ *iff* $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^-)$ *and* $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^+)$ *iff* $1 \in A_S^\perp$.

*Proof.* We argue by induction on the complexity of $A$. The base case is trivial because no atomic sentences are linear tautologies, and if $A$ is an atomic sentence then $\mathtt{e} \notin \mathtt{RSR}(\langle \emptyset, p \rangle)$ in the canonical LL models. And similarly for $A_S^\perp$ and $\mathtt{RSR}(\langle p, \emptyset \rangle)$.

For the induction step, let's start with negation: $\mathtt{e} \in \mathtt{RSR}((\neg\mathtt{a})^-)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^+)$ iff $1 \in A_S^\perp$ iff $1 \in (\neg A)_S$. Similarly, $\mathtt{e} \in \mathtt{RSR}((\neg\mathtt{a})^+)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^-)$ iff $1 \in A_S$ iff $1 \in (\neg A)_S^\perp$.

Multiplicative conjunction: $\mathtt{e} \in \mathtt{RSR}((\mathtt{a} \otimes \mathtt{b})^-)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^-) \sqcup \mathtt{RSR}(\mathtt{b}^-)))$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^+ \sqcup \mathtt{b}^+))$ iff $1 \in (A_S \bullet B_S)^{\perp\perp}$ (by Lemma 126(a)) iff $1 \in (A \otimes B)_S$. Similarly, $\mathtt{e} \in \mathtt{RSR}((\mathtt{a} \otimes \mathtt{b})^+)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^+ \sqcup \mathtt{b}^+)$ iff $1 \in (A_S \bullet B_S)^\perp$ (by Lemma 126(a)) iff $1 \in (A_S \bullet B_S)^{\perp\perp\perp}$ iff $1 \in (A \otimes B)_S^\perp$.

Multiplicative disjunction: $\mathtt{e} \in \mathtt{RSR}((\mathtt{a} \parr \mathtt{b})^-)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^- \sqcup \mathtt{b}^-)$ iff $1 \in (A_S^\perp \bullet B_S^\perp)^\perp$ (by Lemma 126(a)) iff $1 \in (A \parr B)_S$. Similarly $\mathtt{e} \in \mathtt{RSR}((\mathtt{a} \parr \mathtt{b})^+)$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^+) \sqcup \mathtt{RSR}(\mathtt{b}^+)))$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^- \sqcup \mathtt{b}^-))$ iff $1 \in (A_S^\perp \bullet B_S^\perp)^{\perp\perp}$ (by Lemma 126(a)) iff $1 \in (A \parr B)_S^\perp$.

Additive conjunction: $\mathtt{e} \in \mathtt{RSR}((\mathtt{a}\&\mathtt{b})^{-})$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^{-} \sqcap \mathtt{b}^{-})$ iff $1 \in (A_S \cap B_S)$ (by Lemma 126(b)) iff $1 \in (A\&B)_S$. Similarly, $\mathtt{e} \in \mathtt{RSR}((\mathtt{a}\&\mathtt{b})^{+})$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^{+}) \cup \mathtt{RSR}(\mathtt{b}^{+})))$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^{-} \cup \mathtt{b}^{-}))$ iff $1 \in (A_S^{\perp} \cup B_S^{\perp})^{\perp\perp}$ (by Lemma 126(c)) iff $1 \in (\neg A \oplus \neg B)_S$ iff $1 \in (A\&B)_S^{\perp}$.

Additive disjunction: $\mathtt{e} \in \mathtt{RSR}((\mathtt{a} \oplus \mathtt{b})^{-})$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^{-}) \cup \mathtt{RSR}(\mathtt{b}^{-})))$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{RSR}(\mathtt{a}^{+} \cup \mathtt{b}^{+}))$ iff $1 \in (A_S \cup B_S)^{\perp\perp}$ (by Lemma 126(c)) iff $1 \in (A \oplus B)_S$. Similarly, $\mathtt{e} \in \mathtt{RSR}((\mathtt{a} \oplus \mathtt{b})^{+})$ iff $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^{+} \sqcap \mathtt{b}^{+})$ iff $1 \in (A_S^{\perp} \cap B_S^{\perp})$ (by Lemma 126(b)) iff $1 \in (\neg A \& \neg B)_S$ iff $1 \in (A \oplus B)_S^{\perp}$. ■

**Theorem 128.** *A is a linear tautology if and only if* $\overset{\mathcal{M}}{\mid\!\sim}$ *A in all LL implication-space models. And, hence,* $\Gamma \models_{LL} \Delta$ *iff* $\Gamma \overset{LL}{\mid\!\sim} \Delta$.

*Proof.* By Proposition 127, $1 \in A_S$ in all phase structures, which means that $A$ is a linear tautology, if and only if $\mathtt{e} \in \mathtt{RSR}(\mathtt{a}^{-})$ and, hence, $\overset{\mathcal{M}}{\mid\!\sim} A$ in the canonical LL implication-space models. So, $A$ is a linear tautology if and only if $\overset{\mathcal{M}}{\mid\!\sim} A$ in all LL implication-space models. Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ and $\Delta = \{\delta_1, ..., \delta_n\}$. Then $\Gamma \models_{LL} \Delta$ iff $\gamma_1^{\perp} \parr ... \parr \gamma_n^{\perp} \parr \delta_1 \parr ... \parr \delta_m$ is a linear tautology. And the latter holds iff $\overset{\mathcal{M}}{\mid\!\sim} \gamma_1^{\perp} \parr ... \parr \gamma_n^{\perp} \parr \delta_1 \parr ... \parr \delta_m$ in all LL implication-space models, which holds just in case $\Gamma \overset{LL}{\mid\!\sim} \Delta$. ■

### *5.7.6 Implicational Role Inclusions*

We now turn to implicational role inclusions. However, in order to give the proofs of our results regarding role inclusions, we must first establish some connections between implication-space models and strong Kleene valuations.

**Definition 129** (Base Models of a Language $\mathfrak{L}$)**.** Let $\mathfrak{L}_0$ be the set of atomic sentences in $\mathfrak{L}$. A base model of the language $\mathfrak{L}$ is a pair of contents defined by the implication frame, $\langle \mathfrak{L}_0, \mathbb{I} \rangle$, where the interpretation is such that for all $\phi \in \mathfrak{L}$, $[\![\phi]\!] = \langle \mathcal{R}\langle\{\phi\}, \emptyset\rangle, \mathcal{R}\langle\emptyset, \{\phi\}\rangle\rangle$. Base models are monotonic just in case $\forall \mathcal{R}_1, \mathcal{R}_2 \in \mathbb{R}$ (If $\bigcup(\mathcal{R}_1) \subseteq \mathbb{I}$, then $\bigcup((\mathcal{R}_1 \sqcup \mathcal{R}_2)) \subseteq \mathbb{I}$). And they are reflexive just in case $\forall [\![\phi]\!] \in \mathbb{C}\ \forall \mathcal{R} \in \mathbb{R} \bigcup(([\![\phi]\!]^{+} \sqcup [\![\phi]\!]^{-} \sqcup \mathcal{R})) \subseteq \mathbb{I}$.

**Definition 130** (Strong Kleene valuation (model, interpretation))**.** A strong Kleene valuation is a function from sentences of a language $\mathfrak{L}$ to truth-values in $\{0, \frac{1}{2}, 1\}$ that obeys the strong Kleene truth-tables which are, for negation and conjunction (with disjunction and conditionals being defined in the familiar way):

- $v(\neg\phi) = 1 - v(\phi)$
- $v(\phi \wedge \psi) = min(v(\phi), v(\psi))$

**Definition 131** (Maximization of an atomic implication). Let $\langle \Gamma_0, \Delta_0 \rangle$ be a pair of sets of atomic bearers. Then the maximization of it, written $Mx\,\langle \Gamma_0, \Delta_0 \rangle$, is the pair of bearers $\langle \Gamma, \Delta \rangle = \langle \bigcup \Gamma_i, \bigcup \Delta_i \rangle$ for all $\langle \Gamma_i, \Delta_i \rangle$ in the series from $\langle \Gamma_0, \Delta_0 \rangle$ to $\langle \Gamma_\omega, \Delta_\omega \rangle$ inductively defined as follows:

(0) $\Gamma_n \subseteq \Gamma_{n+1}$ and $\Delta_n \subseteq \Delta_{n+1}$.
(i) If $\phi \in \Gamma_n$, then $\neg\phi \in \Delta_{n+1}$. And if $\phi \in \Delta_n$, then $\neg\phi \in \Gamma_{n+1}$.
(ii) If $\phi \in \Gamma_n$ and $\psi \in \Gamma_n$, then $\phi \wedge \psi \in \Gamma_{n+1}$.
(iii) If $\phi \in \Delta_n$ or $\psi \in \Delta_n$, then $\phi \wedge \psi \in \Delta_{n+1}$.

**Lemma 132.** *There is a strong Kleene valuation, $v$, that partitions all sentences of language $\mathfrak{L}$ such that $v(\gamma) = 1$ iff $\gamma \in \Gamma$, and $v(\delta) = 0$ iff $\delta \in \Delta$, and $v(\theta) = \frac{1}{2}$ iff $\theta \in \Theta$, if and only if there is a monotonic and reflexive base model of $\mathfrak{L}$, such that $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta \rangle = Mx\,\langle \Gamma_0, \Delta_0 \rangle$.*

*Proof.* Left-to-right: Suppose that there is a strong Kleene valuation that partitions all sentences such that $v(\gamma) = 1$ iff $\gamma \in \Gamma$, and $v(\delta) = 0$ iff $\delta \in \Delta$, and $v(\theta) = \frac{1}{2}$ iff $\theta \in \Theta$. Let $\Gamma_0 = \Gamma \cap \mathfrak{L}_0$ be the atomic sentences in $\Gamma$, and let $\Delta_0 = \Delta \cap \mathfrak{L}_0$ be the atomic sentences in $\Delta$. The base model in which $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ is the desired model. To see this, note that since $\Gamma_0 \cap \Delta_0 = \varnothing$, the stipulation that $\{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\} \not\subseteq \mathbb{I}$ does not conflict with the reflexivity of the base model. Moreover, suppose that $\bigcup(\mathcal{R}_1) \subseteq \mathbb{I}$. So, $\bigcup(\mathcal{R}_1) \cap \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\} = \varnothing$. But then for any $\mathcal{R}_2$ the adjunction $\bigcup(\mathcal{R}_1 \sqcup \mathcal{R}_2) \cap \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\} = \varnothing$. So, $\bigcup(\mathcal{R}_1 \sqcup \mathcal{R}_2) \subseteq \mathbb{I}$. Hence, $\forall \mathcal{R}_1, \mathcal{R}_2 \in \mathbb{R}$ (if $\bigcup(\mathcal{R}_1) \subseteq \mathbb{I}$, then $\bigcup(\mathcal{R}_1 \sqcup \mathcal{R}_2) \subseteq \mathbb{I}$), and so our stipulation implies that our base model is monotonic.

To show that $\langle \Gamma, \Delta \rangle = Mx\,\langle \Gamma_0, \Delta_0 \rangle$, we argue by induction on steps in our construction of $Mx\,\langle \Gamma_0, \Delta_0 \rangle$, namely that for all sentences, $\phi_n$, of complexity $n$, we have $v(\phi_n) = 1$ iff $\phi_n \in \Gamma_n$, and $v(\phi_n) = 0$ iff $\phi_n \in \Delta_n$. All the other sentences of any complexity must then be in $\Theta$. Base case: For $\langle \Gamma_0, \Delta_0 \rangle$ it is immediate that it is the pair of atomic sentences, i.e., sentences of complexity 0, from $\Gamma$ and $\Delta$. So that, $v(\phi_0) = 1$ iff $\phi_0 \in \Gamma_0$, and $v(\phi_0) = 0$ iff $\phi_0 \in \Delta_0$.

For the induction step, take a sentence, $\phi$, of complexity, $n + 1$. If $\phi_{n+1}$ is either in $\Gamma$ or in $\Delta$, then it is added at step $n + 1$ of the construction of $Mx\,\langle \Gamma_0, \Delta_0 \rangle$. And $\phi$ is either a negation or a conjunction. Suppose $\phi = \neg\psi$. Now, $\neg\psi \in \Gamma_{n+1}$ iff $\psi \in \Delta_n$, which happens, by our induction hypothesis iff $v(\psi) = 0$, which by the truth-tables happens iff $v(\neg\psi) = 1$. Similarly, $\neg\psi \in \Delta_{n+1}$ iff $\psi \in \Gamma_n$, which happens, by our induction hypothesis iff

$v(\psi) = 1$, which by the truth-tables happens iff $v(\neg\psi) = 0$. Next, suppose that $\phi = \psi \wedge \chi$. Now, $\psi \wedge \chi \in \Gamma_{n+1}$ iff $\psi \in \Gamma_n$ and $\chi \in \Gamma_n$, which happens, by our induction hypothesis iff $v(\psi) = 1$ and $v(\chi) = 1$, which by the truth-tables happens iff $v(\psi \wedge \chi) = 1$. Similarly, $\psi \wedge \chi \in \Delta_{n+1}$ iff $\psi \in \Delta_n$ or $\chi \in \Delta_n$, which happens, by our induction hypothesis iff $v(\psi) = 0$ or $v(\chi) = 0$, which by the truth-tables happens iff $v(\psi \wedge \chi) = 0$. So, by induction, $\langle \Gamma, \Delta \rangle = Mx \langle \Gamma_0, \Delta_0 \rangle$.

Right-to-left: Suppose that there is a monotonic and reflexive base model of $\mathfrak{L}$, such that $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta \rangle = Mx \langle \Gamma_0, \Delta_0 \rangle$. Take the valuation such that $v(\gamma_0) = 1$ iff $\gamma_0 \in \Gamma_0$, and $v(\delta_0) = 0$ iff $\delta_0 \in \Delta_0$, and $v(\theta_0) = \frac{1}{2}$ iff $\theta_0 \in \mathfrak{L}_0$ but $\theta_0 \notin \Gamma_0 \cup \Delta_0$. Then we use the strong Kleene truth-tables to determine the values of all logically complex sentences. The result is a strong Kleene valuation, $v$, that partitions all sentences of language $\mathfrak{L}$ such that $v(\gamma) = 1$ iff $\gamma \in \Gamma$, and $v(\delta) = 0$ iff $\delta \in \Delta$, and $v(\theta) = \frac{1}{2}$ iff $\theta \in \Theta$. This can be shown by an induction on the steps in the construction of $Mx \langle \Gamma_0, \Delta_0 \rangle$, which is analogous to the induction for the left-to-right direction. ■

**Lemma 133.** *For every set $\mathscr{I} \subseteq \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0)$, there is a monotonic and reflexive base model of $\mathfrak{L}$, such that $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta \rangle = Mx \langle \Gamma_0, \Delta_0 \rangle$ and $\mathscr{I} = \{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$ if and only if there is a strong Kleene valuation, $v$, that is an* ST *counterexample to all and only the inferences in $\mathscr{I}$.*

*Proof.* Left-to-right: For all candidate implications $\langle X, Y \rangle \in \mathscr{I}$, we know that $X \subseteq \Gamma$ and $Y \subseteq \Delta$. By Lemma 132, there is a strong Kleene valuation such that $v(\gamma) = 1$ iff $\gamma \in \Gamma$, and $v(\delta) = 0$ iff $\delta \in \Delta$, and $v(\theta) = \frac{1}{2}$ iff $\theta \in \Theta$. Hence, for all $\langle X, Y \rangle \in \mathscr{I}, \gamma \in X(v(\gamma) = 1)$ and $\forall \delta \in Y(v(\delta) = 0)$. Hence, $v$ is an ST counterexample to everything in $\mathscr{I}$.

Right-to-left: Suppose that there is a strong Kleene valuation, $v$, that is an ST counterexample to all and only the inferences in $\mathscr{I}$. By Lemma 132, there is a monotonic and reflexive base model of $\mathfrak{L}$, such that candidate implication $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $Mx(\langle \Gamma_0, \Delta_0 \rangle) = \langle \Gamma, \Delta \rangle$ such that $\Gamma$ is the exact set of sentences with value 1 in $v$ and $\Delta$ is the exact set of the sentences with value 0 in $v$. Hence, the inferences to which $v$ provides ST counterexamples are all and only the inferences, $X \succ Y$, such that $X \subseteq \Gamma$ and $Y \subseteq \Delta$. That is, $\mathscr{I} = \{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$. ■

According to the definition in the main text, a conic implication-space model, $\mathcal{M}$, is a monotonic and reflexive implication-space model in which there is a pair $\langle \Gamma, \Delta \rangle$ and $X \not\mid\!\sim^{\mathcal{M}} Y$ iff $\langle X, Y \rangle \in \mathcal{P}(\Gamma) \times \mathcal{P}(\Delta)$.

**Proposition 134.** *There is a monotonic and reflexive base model of* $\mathfrak{L}$, *such that* $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ *and* $\langle \Gamma, \Delta \rangle = Mx\, \langle \Gamma_0, \Delta_0 \rangle$ *if and only if there is a conic implication-space model that is a counterexample to all and only the candidate implications in the set* $\{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$.

*Proof.* Left-to-right: Suppose that there is a monotonic and reflexive base model of $\mathfrak{L}$, such that $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta \rangle = Mx\, \langle \Gamma_0, \Delta_0 \rangle$. We know that $\langle \Gamma_0, \Delta_0 \rangle \notin \mathbb{I}$. But by the construction of $Mx\, \langle \Gamma_0, \Delta_0 \rangle$, if $\langle \Gamma_i, \Delta_i \rangle \notin \mathbb{I}$, then $\langle \Gamma_{i+1}, \Delta_{i+1} \rangle \notin \mathbb{I}$. Therefore, $\langle \Gamma, \Delta \rangle \notin \mathbb{I}$. And by monotonicity, this implies that none of the implications in the set $\{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$ are in $\mathbb{I}$. Conversely, suppose that $\langle U, W \rangle \notin \mathbb{I}$. If the sentences in $U$ and $W$ are atomic, then $\langle U, W \rangle \in \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$. And if they are not atomic, we can reduce them to atomic sequents by applying the procedure by which we constructed $Mx\, \langle \Gamma_0, \Delta_0 \rangle$ in reverse. The resulting sequents must be in $\{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$. And from the construction of $Mx\, \langle \Gamma_0, \Delta_0 \rangle$ we can then see that $\langle U, W \rangle \in \{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$. So our model is a counterexample to all and only the candidate implications in the set $\{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$.

Right-to-left: Suppose that there is a monotonic and reflexive implication-space model that is a counterexample to all and only the candidate implications in the set $\{\langle X, Y \rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$. We can now construct a model, $\mathcal{M}'$, by taking the implication frame defined by the set of atomic sentences as bearers and the consequence relation of our implication-space model over these atomic sentences as the set of good implications, and it is easy to see that the consequence relation of this base model is a monotonic and reflexive base model of $\mathfrak{L}$, such that $\mathbb{I}_{\mathcal{M}'} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y \rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta \rangle = Mx\, \langle \Gamma_0, \Delta_0 \rangle$. ■

**Theorem 135.** *There is a strong Kleene model that is an* ST *counterexample to all and only the candidate implications in a set* $\mathscr{K}$ *if and only if there is a conic implication-space model that is a counterexample to all and only the candidate implications in a set* $\mathscr{K}$.

*Proof.* Left-to-right: There is a strong Kleene model, $v$, that is an ST counterexample to all and only the candidate implications in a set $\mathscr{K}$ iff for all and only the implications $\langle P, C \rangle \in \mathscr{K}$ we have $v(p) = 1$ for all $p \in P$, and $v(c) = 0$ for all $c \in C$. Let $\Theta$ be $\bigcup P$ for all $\langle P, C \rangle \in \mathscr{K}$, and let $\Lambda$ be $\bigcup C$ for all $\langle P, C \rangle \in \mathscr{K}$. The counterexample in question is one that partitions all sentences of language $\mathfrak{L}$ such that $v(\theta) = 1$ iff $\theta \in \Theta$, and $v(\lambda) = 0$ iff $\lambda \in \Lambda$. By Lemma 132, there is a strong Kleene valuation, $v$, that partitions all sentences of language $\mathfrak{L}$ such that $v(\theta) = 1$ iff $\theta \in \Theta$, and $v(\lambda) = 0$ iff $\lambda \in \Lambda$, if and only if there is a monotonic and reflexive base model of $\mathfrak{L}$, such that candidate implication $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y\rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta\rangle = Mx\,\langle \Gamma_0, \Delta_0\rangle$. And from Lemma 133 we know that the bad candidate implications in such a model are $\mathscr{K} = \{\langle X, Y\rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$. Hence, there is a conic implication-space model that is a counterexample to all and only the candidate implications in a set $\mathscr{K}$.

Right-to-left: Suppose that there is a conic implication-space model that is a counterexample to all and only the candidate implications in a set $\mathscr{K}$. The set of all bad candidate implications is $\mathscr{K} = \{\langle X, Y\rangle \mid X \subseteq \Gamma, Y \subseteq \Delta\}$ for some $\langle \Gamma, \Delta\rangle$. By Proposition 134, there is such a conic implication-space model iff there is a monotonic and reflexive base model of $\mathfrak{L}$, such that $\mathbb{I} = \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0) \setminus \{\langle X, Y\rangle \mid X \subseteq \Gamma_0, Y \subseteq \Delta_0\}$ and $\langle \Gamma, \Delta\rangle = Mx\,\langle \Gamma_0, \Delta_0\rangle$. And by Lemma 132, this happens just in case there is a strong Kleene model that is an ST counterexample to all and only the candidate implications in a set $\mathscr{K}$. ■

**Definition 136** (Local metainferential validity). A metainference from the set of sequents $X$ to the set of sequents $Y$ is valid in the local sense if and only if, for every model (valuation) (in the relevant class of models), if the model is a counterexample to all of the sequents in $Y$, then it is a counterexample to at least one of the sequents in $X$.

**Fact 137.** *The following are implications of well known facts about the relations between* ST, LP, K3, *and* TS *(see Dicher and Paoli, 2019; Barrio et al., 2015; Cobreros et al., 2020a; Ré et al., 2021):*

$\Gamma \models_{LP} \Delta$ *iff the metainference from* $\succ \bigwedge \Gamma$ *to* $\succ \Delta$ *is locally valid in the strong Kleene formulation of* ST.

$\Gamma \models_{K3} \Delta$ *iff the metainference from* $\bigvee \Delta \succ$ *to* $\Gamma \succ$ *is locally valid in the strong Kleene formulation of* ST.

$\Gamma \models_{ST} \Delta$ *iff the metainference from* $A \succ A$ *to* $\Gamma \succ \Delta$ *is locally valid in the strong Kleene formulation of* ST.

$\Gamma \models_{TS} \Delta$ *iff the metainference from* $\{\succ \bigwedge \Gamma, \bigvee \Delta \succ\}$ *to* $\emptyset \succ \emptyset$ *is locally valid in the strong Kleene formulation of* ST.

*Proof.* One way to see why these facts hold is the following: A strong Kleene counterexample to $\Gamma \models_{LP} \Delta$ is such that $\forall \gamma \in \Gamma\ (v(\gamma) \neq 0)$ and $\forall \delta \in \Delta\ (v(\delta) = 0)$. But a strong Kleene model is an ST counterexample to $\succ \bigwedge \Gamma$ iff $\exists \gamma \in \Gamma\ (v(\gamma) = 0)$. So the models such that $\forall \gamma \in \Gamma\ (v(\gamma) \neq 0)$ are those that could be counterexamples to the metainference or the LP inference, if we considered just $\Gamma$. A counterexample to $\succ \Delta$ is a model such that $\forall \delta \in \Delta\ (v(\delta) = 0)$. So a model is a counterexample to the metainference just in case it is a counterexample to $\Gamma \models_{LP} \Delta$.

Similarly, a strong Kleene counterexample to $\Gamma \models_{K3} \Delta$ is such that $\forall \gamma \in \Gamma\ (v(\gamma) = 1)$ and $\forall \delta \in \Delta\ (v(\delta) \neq 1)$. But a strong Kleene model is an ST counterexample to $\bigvee \Delta \succ$ iff $\exists \delta \in \Delta\ (v(\delta) = 1)$. So the models such that $\forall \delta \in \Delta\ (v(\delta) \neq 1)$ are those that could be counterexamples to the metainference or the K3 inference, if we considered just $\Delta$. A counterexample to $\Gamma \succ$ is a model such that $\forall \gamma \in \Gamma\ (v(\gamma) = 1)$. So a model is a counterexample to the metainference just in case it is a counterexample to $\Gamma \models_{K3} \Delta$.

Of course, $\Gamma \models_{ST} \Delta$ holds iff there is no ST counterexample to $\Gamma \succ \Delta$. Since there isn't any countermodel to the inference $A \succ A$ in the strong Kleene formulation of ST, this holds just in case the metainference from $A \succ A$ to $\Gamma \succ \Delta$ is locally valid in the strong Kleene formulation of ST.

A strong Kleene counterexample to $\Gamma \models_{TS} \Delta$ is a model in which $\forall \gamma \in \Gamma\ (v(\gamma) \neq 0)$ and $\forall \delta \in \Delta\ (v(\delta) \neq 1)$. So $\Gamma \models_{TS} \Delta$ iff every model is such that $\exists \gamma \in \Gamma\ (v(\gamma) = 0)$ or $\exists \delta \in \Delta\ (v(\delta) = 1)$. That is, $\Gamma \models_{TS} \Delta$ iff every model is an ST counterexample to $\succ \bigwedge \Gamma$ or to $\bigvee \Delta \succ$. Since all strong Kleene models are counterexamples to $\emptyset \succ \emptyset$, this holds just in case the metainference from $\{\succ \bigwedge \Gamma,\ \bigvee \Delta \succ\}$ to $\emptyset \succ \emptyset$ is locally valid in the strong Kleene formulation of ST. ∎

**Lemma 138.** *A metainference from* $\{\Theta_1 \succ \Lambda_1, ..., \Theta_n \succ \Lambda_n\}$ *to* $\{\Gamma \succ \Delta\}$ *is locally valid in the strong Kleene formulation of* ST *just in case* $\mathcal{R}\langle\Theta_1, \Lambda_1\rangle, ..., \mathcal{R}\langle\Theta_n, \Lambda_n\rangle \preceq \mathcal{R}\langle\Gamma, \Delta\rangle$ *in the conic implication-space models.*

*Proof.* Left-to-right: Suppose that the metainference from $\{\Theta_1 \succ \Lambda_1, ..., \Theta_n \succ \Lambda_n\}$ to $\{\Gamma \succ \Delta\}$ is locally valid in the strong Kleene formulation of ST. By Theorem 135 above, there is a one-to-one mapping between conic implication-space models and ST models that preserves counterexamples in both directions. Hence, every conic implication-space model that is a counterexample to $\Gamma \succ \Delta$ is a counterexample to at least one of the implications in $\{\Theta_1 \succ \Lambda_1, ..., \Theta_n \succ \Lambda_n\}$. So, in all conic implication-space models, if $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Gamma, \Delta\rangle)$, then $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Theta_1, \Lambda_1\rangle)$ or ... or $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Theta_n, \Lambda_n\rangle)$, i.e., $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Theta_1, \Lambda_1\rangle \sqcap ... \sqcap \mathcal{R}\langle\Theta_n, \Lambda_n\rangle)$. In other words, it holds just in case: in all models, if $\mathtt{e} \in \mathtt{RSR}(\mathcal{R}\langle\Theta_1, \Lambda_1\rangle \sqcap ... \sqcap \mathcal{R}\langle\Theta_n, \Lambda_n\rangle)$, then $\mathtt{e} \in \mathtt{RSR}(\mathcal{R}\langle\Gamma, \Delta\rangle)$. Since we can vary $\mathbb{I}$ in any way that is consistent with the model being conic, this holds just in case, in all conic models, if $x \in \mathtt{RSR}(\mathcal{R}\langle\Theta_1, \Lambda_1\rangle \sqcap ... \sqcap \mathcal{R}\langle\Theta_n, \Lambda_n\rangle)$, then $x \in \mathtt{RSR}(\mathcal{R}\langle\Gamma, \Delta\rangle)$, that is, $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}\langle\Theta_i, \Lambda_i\rangle) \subseteq \mathtt{RSR}(\bigsqcup_{j=k}^{m} \mathcal{R}\langle\Gamma, \Delta\rangle)$. By definition, this is $\mathcal{R}\langle\Theta_1, \Lambda_1\rangle, ..., \mathcal{R}\langle\Theta_n, \Lambda_n\rangle \preceq \mathcal{R}\langle\Gamma, \Delta\rangle$ in the conic models.

Right-to-left: Suppose that $\mathcal{R}\langle\Theta_1,\Lambda_1\rangle,...,\mathcal{R}\langle\Theta_n,\Lambda_n\rangle \preceq \mathcal{R}\langle\Gamma,\Delta\rangle$ in the conic implication-space models. So, in all conic implication-space models, if $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Gamma,\Delta\rangle)$, then $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Theta_1,\Lambda_1\rangle)$ or ... or $\mathtt{e} \notin \mathtt{RSR}(\mathcal{R}\langle\Theta_n,\Lambda_n\rangle)$. By Theorem 135 above, it follows that the metainference from $\{\Theta_1 \succ \Lambda_1,...,\Theta_n \succ \Lambda_n\}$ to $\{\Gamma \succ \Delta\}$ is locally valid in the strong Kleene formulation of ST. ■

**Proposition 139.** *Suppose that the only constraint on implication-space models is that they be conic, and let* $[\![\Gamma]\!] = \mathtt{G}$ *and* $\mathtt{G}^+ = \{\mathtt{g}^+ \mid \mathtt{g} \in \mathtt{G}\}$ *and* $\mathtt{G}^- = \{\mathtt{g}^- \mid \mathtt{g} \in \mathtt{G}\}$ *and analogously for* $[\![\Delta]\!] = \mathtt{D}$. *Then:*
$\Gamma \models_{LP} \Delta$ *if and only if* $(\bigwedge \mathtt{G})^- \preceq \mathtt{D}^-$.
$\Gamma \models_{K3} \Delta$ *if and only if* $(\bigvee \mathtt{D})^+ \preceq \mathtt{G}^+$.
$\Gamma \models_{ST} \Delta$ *if and only if* $\star \preceq \mathtt{G}^+, \mathtt{D}^-$.
$\Gamma \models_{TS} \Delta$ *if and only if* $(\bigwedge \mathtt{G})^-, (\bigvee \mathtt{D})^+ \preceq \mathtt{e}$.

*Proof.* This follows from Fact 137 and Lemma 138. ■

We now turn to the relation between implication-space semantics and work related to analytic metaphysics, in particular Correia's logic of factual equivalence and the notions of essence, grounding, and generalized identity.

**Definition 140** (Essential truth-maker models)**.** An essential truth-maker model of a truth-maker model $\mathcal{M}$ is a truth-maker model $\mathcal{M}'$ that is like $\mathcal{M}$ in its consequence relations but in which every truth-maker of $A$ is a truth-maker of $B$ if, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma, A \models_{\overline{TM}} \Delta$, then $\Gamma, B \models_{\overline{TM}} \Delta$, and every falsity-maker of $A$ is a falsity-maker of $B$ if, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma \models_{\overline{TM}} B, \Delta$, then $\Gamma \models_{\overline{TM}} A, \Delta$.

Essential truth-maker models, in effect, are the result of ignoring all the differences between states in a truth-maker model that do not matter for the possibility or impossibility of states.

**Lemma 141.** *There is a truth-maker model such that* $\models_{\overline{TM}}= X$ *in that model if and only if there is an essential truth-maker model such that* $\models_{\overline{TM}}= X$ *in that essential model.*

*Proof.* The right-to-left direction is immediate. For the left-to-right direction, take a truth-maker model $\mathcal{M}$ and add the following stipulation to it: for any state $w$ that is a fusion of truth-makers for each sentence in a set $\Gamma$ and of falsity-makers for each sentence in a set $\Delta$, for all sets of states $T$ and $S$, if (if $\forall t \in T(t \Cup w \notin S^\diamond)$, then $\forall s \in S(s \Cup w \notin S^\diamond)$), then $S \subseteq T$. This obviously does not change the consequence relation of $\mathcal{M}$. And the resulting model is an essential model of $\mathcal{M}$. To see this, suppose that, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma, A \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$, then $\Gamma, B \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$. Take any state $w$ and let $\langle\Gamma,\Delta\rangle$ be such that $w$ is a fusion of truth-makers for every sentence in $\Gamma$ and falsity-makers for every sentence in $\Delta$. Since if $\Gamma, A \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$, then $\Gamma, B \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$, it follows that if $\forall t \in |A|^+ \, (t \Cup w \notin S^\diamond)$, then $\forall s \in |B|^+ \, (s \Cup w \notin S^\diamond)$. And since $w$ was arbitrary our stipulation implies that $|B|^+ \subseteq |A|^+$. For the condition regarding falsity-makers, suppose that, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma \mathrel{|\!\!\underset{TM}{\equiv}} B, \Delta$, then $\Gamma \mathrel{|\!\!\underset{TM}{\equiv}} A, \Delta$. Then, for any $w$, if $\forall t \in |A|^+ \, (t \Cup w \notin S^\diamond)$, then $\forall s \in |B|^+ \, (s \Cup w \notin S^\diamond)$. So $|B|^- \subseteq |A|^-$. ■

This means that the nonessential truth-maker models are, in one sense, superfluous: Truth-maker theory can codify a consequence relation without the nonessential models just in case it can codify the consequence relation with the nonessential models. That is why we call these truth-maker models nonessential and the others essential truth-maker models.

**Lemma 142.** *Let $[\![A]\!] = \mathtt{a}$ and $[\![B]\!] = \mathtt{b}$. We have $\mathtt{a}^+ \preceq \mathtt{b}^+$ in all implication-space models in a set $m$, if and only if, in all the essential truth-maker models of truth-maker models whose consequence relation coincides with a consequence relation of a model in $m$, every truth-maker of $A$ is a truth-maker of $B$. And $\mathtt{a}^- \preceq \mathtt{b}^-$ in all implication-space models in a set $m$, if and only if every falsity-maker of $A$ is a falsity-maker of $B$ in the corresponding truth-maker models.*

*Proof.* Left-to-right: We know from Theorem 83 that for every implication-space model, there is a truth-maker model with the same consequence relation, and vice versa. Suppose that $\mathtt{a}^+ \preceq \mathtt{b}^+$ in all implication-space models in $m$, that is, $\mathtt{RSR}(\mathtt{a}^+) \subseteq \mathtt{RSR}(\mathtt{b}^+)$. So, in all these models, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma, A \mathrel{|\!\!\sim} \Delta$, then $\Gamma, B \mathrel{|\!\!\sim} \Delta$. Hence, in all the corresponding truth-maker models, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma, A \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$, then $\Gamma, B \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$. In an essential truth-maker model, this ensures that every truth-maker of $A$ is a truth-maker of $B$. Suppose that $\mathtt{a}^- \preceq \mathtt{b}^-$ in all implication-space models in $m$, that is, $\mathtt{RSR}(\mathtt{a}^-) \subseteq \mathtt{RSR}(\mathtt{b}^-)$. So, in all these models, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma \mathrel{|\!\!\sim} A, \Delta$, then $\Gamma \mathrel{|\!\!\sim} B, \Delta$. Hence, in all the corresponding truth-maker models, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma \mathrel{|\!\!\underset{TM}{\equiv}} A, \Delta$, then $\Gamma \mathrel{|\!\!\underset{TM}{\equiv}} B, \Delta$. In an essential truth-maker model, this ensures that every falsity-maker of $A$ is a falsity-maker of $B$ in the corresponding truth-maker models.

Suppose that in all the essential truth-maker models for the models $m$, every truth-maker of $A$ is a truth-maker of $B$. Hence, in all these models, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma, A \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$, then $\Gamma, B \mathrel{|\!\!\underset{TM}{\equiv}} \Delta$. So, in all the implication-space models, for all $\langle\Gamma,\Delta\rangle$, if $\Gamma, A \mathrel{|\!\!\sim} \Delta$, then $\Gamma, B \mathrel{|\!\!\sim} \Delta$. Therefore, $\mathtt{a}^+ \preceq \mathtt{b}^+$. And analogous reasoning shows that if every falsity-maker of $A$ is a falsity-maker of $B$ in the corresponding essential truth-maker models, then $\mathtt{a}^- \preceq \mathtt{b}^-$ in all implication-space models in a set $m$. ■

**Proposition 143.** *If the only constraint on models is that they obey the semantic clauses for the implication-space semantic of* NMMS *(Definition 70), then Correia's logic and his dual system are related to implicational roles as follows. For all* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*:*

$\vdash_c A \approx B$ *if and only if, in all models,* $\mathtt{a}^+ \preceq\succeq \mathtt{b}^+$*, and so* $\mathtt{a}^+ = \mathtt{b}^+$.

$\vdash_{dc} A \approx B$ *if and only if, in all models,* $\mathtt{a}^- \preceq\succeq \mathtt{b}^-$*, and so* $\mathtt{a}^- = \mathtt{b}^-$.

*Proof.* Correia uses, in effect, the same truth-maker models that we use for NMMS.[26] Moreover, Correia (2016) has shown that $\vdash_c A \approx B$ holds just in case $A$ and $B$ have the same truth-makers in all truth-maker models (and so merely in virtue of logical form), and it is easy to see that $\vdash_{dc} A \approx B$ holds just in case $A$ and $B$ have the same falsity-makers in all truth-maker models (see Hlobil, 2022a). By Lemma 141, something holds in all truth-maker models iff it holds in all essential truth-maker models. And by Lemma 142, $A$ and $B$ have the same truth-makers in all essential truth-maker models iff $\mathtt{a}^+ \preceq\succeq \mathtt{b}^+$. And $A$ and $B$ have the same falsity-makers in all essential truth-maker models iff $\mathtt{a}^- \preceq\succeq \mathtt{b}^-$. ■

**Fact 144.** *Factual Essence: Its being the case that A is what it is for it to be the case that B in full, in virtue of truth-functional logical form (Correia and Skiles, 2019, 651), if and only if, for* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*, we have* $\mathtt{a}^+ = \mathtt{b}^+$ *in all implication-space models. And its being the case that A is in part what it is for it to be the case that B, in virtue of truth-functional logical form (Correia and Skiles, 2019, 651), if and only if, for* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*, there is some* $\mathtt{c}^+$ *such that* $\mathtt{a}^+ \sqcup \mathtt{c}^+ = \mathtt{b}^+$ *in all implication-space models.*

*Proof.* Correia and Skiles (2019, 651) use Correia's logic to suggest that: its being the case that $A$ is what it is for it to be the case that $B$ in full, in virtue of truth-functional logical form, if and only if, $\vdash_c A \approx B$ in virtue of truth-functional logical form. We know from Proposition 143 that $\vdash_c A \approx B$ holds in virtue of truth-functional logical form just in case, for $[\![A]\!] = \mathtt{a}$ and $[\![B]\!] = \mathtt{b}$, we have $\mathtt{a}^+ = \mathtt{b}^+$ in all implication-space models.

Correia and Skiles (2019, 651) also use Correia's logic to suggest that: Its being the case that $A$ is in part what it is for it to be the case that $B$ iff: there is some $C$ such that its being the case that $B$ is for it to be the case that both $A$ and $C$. Using Correia's logic, this holds in virtue of truth-functional logical form just in case $\vdash_c B \approx A \wedge C$. It follows from Proposition 143 that $\vdash_c B \approx A \wedge C$ holds just in case, for $[\![A]\!] = \mathtt{a}$ and $[\![B]\!] = \mathtt{b}$ and $[\![C]\!] = \mathtt{c}$, we have $\mathtt{a}^+ \sqcup \mathtt{c}^+ = \mathtt{b}^+$ in all implication-space models. So, its being the case that $A$ is in part what it is for it to be the case that $B$ holds, in virtue of truth-functional logical form, iff there is some $\mathtt{c}^+$ such that $\mathtt{a}^+ \sqcup \mathtt{c}^+ = \mathtt{b}^+$ in all implication-space models. ■

**Fact 145.** *Strict Full Grounding: Its being the case that* $A_1, \ldots, A_n$ *makes it the case that B, in virtue of truth-functional logical form (Correia and Skiles, 2019, 655), if and only if, in all implication-space models, for* $[\![A_i]\!] = \mathtt{a}_i$ *and* $[\![B]\!] = \mathtt{b}$*, (i) for some* $[\![C]\!] = \mathtt{c}$*, we have* $(\mathtt{a}_1^+ \sqcup \ldots \sqcup \mathtt{a}_n^+) \sqcap \mathtt{c}^+ \sqcap (\mathtt{a}_1^+ \sqcup \ldots \sqcup \mathtt{a}_n^+ \sqcup \mathtt{c}^+) \preceq\succeq \mathtt{b}^+$ *and (ii)* $\forall 1 \leq i \leq n$ *there are no* $[\![D]\!] = \mathtt{d}$ *and* $[\![E]\!] = \mathtt{e}$ *such that* $(\mathtt{b}^+ \sqcup \mathtt{d}^+) \sqcap \mathtt{e}^+ \sqcap (\mathtt{b}^+ \sqcup \mathtt{d}^+ \sqcup \mathtt{e}^+) \preceq\succeq \mathtt{a}_i^+$.

*Proof.* Correia and Skiles (2019, 655) suggest, using Correia's logic, that its being the case that $A_1, \ldots, A_n$ makes it the case that $B$ (in the sense of strict full grounding), in virtue of truth-functional logical form, iff (i) for some $C$, we have $\vdash_c (A_1 \wedge \ldots \wedge A_n) \vee C \approx B$ and (ii) neither is there a $D$ such that $\exists E(\vdash_c (B \wedge D) \vee E \approx A_1)$ nor ... nor is there a $D$ such that $\exists E(\vdash_c (B \wedge D) \vee E \approx A_n)$. It follows from Proposition 143 that condition (i) holds iff for some $C$, with $[\![C]\!] = \mathtt{c}$, we have$(\mathtt{a}_1^+ \sqcup \ldots \sqcup \mathtt{a}_n^+) \sqcap \mathtt{c}^+ \sqcap (\mathtt{a}_1^+ \sqcup \ldots \sqcup \mathtt{a}_n^+ \sqcup \mathtt{c}^+) \preceq\succeq \mathtt{b}^+$ in all implication-space models. And condition (ii) holds iff $\forall 1 \leq i \leq n$ there are no $[\![D]\!] = \mathtt{d}$ and $[\![E]\!] = \mathtt{e}$ such that $(\mathtt{b}^+ \sqcup \mathtt{d}^+) \sqcap \mathtt{e}^+ \sqcap (\mathtt{b}^+ \sqcup \mathtt{d}^+ \sqcup \mathtt{e}^+) \preceq\succeq \mathtt{a}_i^+$. ■

**Fact 146.** *A sentence of the form "For it to be the case that A is for it to be the case that B" is true in virtue of logical form, in the sense of Elgin (2021, sec 4), if and only if, for* $[\![A]\!] = \mathtt{a}$ *and* $[\![B]\!] = \mathtt{b}$*, we have* $\mathtt{a} = \mathtt{b}$*, and so* $\mathtt{a} \ni\in \mathtt{b}$*, in all implication-space models.*

*Proof.* Elgin holds that two sentences are exactly equivalent iff they have the same truth-makers and the same falsity-makers. He illustrated his suggestion for generalized identity thus: "'To be a person is to be bound by the categorical imperative' holds just in case the verifiers and falsifiers of the claim that someone is a person are identical to the verifiers and falsifiers of the claim that she is bound by the categorical imperative" (Elgin, 2021, 9). If we think of sentences as 0-place predicates, this yields the claim that a sentence of the form "For it to be the case that $A$ is for it to be the case that $B$" holds iff $A$ and $B$ have the same truth-makers and the same falsity-makers. And it holds in virtue of logical form if it holds in all truth-maker models. Now, if $A$ and $B$ have the same truth-makers and the same falsity-makers in all truth-maker models, then this holds in all essential truth-maker models. Hence, $\mathtt{a} = \mathtt{b}$, and so $\mathtt{a} \ni\in \mathtt{b}$, in all implication space models, for $[\![A]\!] = \mathtt{a}$ and $[\![B]\!] = \mathtt{b}$. ■

The facts stated in the section on nonlogical role inclusions are all immediate implications of the correspondences between sequent rules and implication-space semantics, which we highlight in various places in the chapter, especially in the subsection on the implication-space semantics of NMMS and in the subsection on linear logic.

## Notes

1 Implication-space semantics is closely related to Girard's (1987) phase-space semantics for linear logic. The formalism that we present in this chapter is inspired by and similar to the one developed in Daniel Kaplan's (2022) PhD dissertation. However, our definitions of implicational roles and adjunction are different from Kaplan's definitions. As a result our semantic clauses for the logical vocabulary are also different. Unlike Kaplan, we define implicational roles by abstraction, using an equivalence relation among candidate implications.

2 Note that we do not ignore incompatibilities here. Rather, as in previous chapters, we encode implications and incompatibilities in a single reason relation, namely by using the empty set as a conclusion set in a good implication to encode the mutual incompatibility of the premises. For ease of exposition we sometimes call the combined relation an implication relation; but it should be kept in mind that this relation also includes all information about material incompatibilities.

3 Recall from Chapter Three that not just good implications but also mere candidate implications have ranges of subjunctive robustness, that is, sets of additions that yield good implications. Hence, $\langle \{\phi\}, \emptyset \rangle$ need not be a good implication to have a range of subjunctive robustness.

4 We will allow ourselves to write $\mathtt{RSR}\,\langle X, Y\rangle$ for $\mathtt{RSR}(\langle X, Y\rangle)$, wherever this seems appropriate to avoid clutter.

5 We will allow ourselves to write $\mathcal{R}\,\langle X, Y\rangle$ for $\mathcal{R}(\langle X, Y\rangle)$, wherever this seems appropriate to avoid clutter.

6 The case of the role of an arbitrary candidate implication $\langle \Theta, \Lambda\rangle$ is analogous. Again, the role of a candidate implication is the set of sets of candidate implications that have the same range of subjunctive robustness: the set of sets of candidate implications such that, say, $\{\langle \Gamma, \Delta\rangle\}$ is in that set if and only if parallel additions of premises or conclusions to $\langle \Gamma, \Delta\rangle$ and to $\langle \Theta, \Lambda\rangle$ always either both yield good implications or both yield implications that are not good. Again, the idea is that an implicational role is an equivalence class, where the relevant equivalence relation is the relation of having the same range of subjunctive robustness. And we can also express this by saying that if $\langle \Gamma, \Delta\rangle$ and $\langle \Theta, \Lambda\rangle$ have the same implicational role, then they can be substituted for each other *salva consequentia*.

7 Cantor's Theorem ensures that, in every implication space, the cardinality of the bearers is always smaller than the cardinality of the potential conceptual contents in that implication space. After all, every set of candidate implications can serve as a range of subjunctive robustness that can define a premisory or conclusory role. So $|\mathcal{P}(\mathbb{S})|$, which is $|\mathcal{P}(\mathcal{P}(\mathtt{B}) \times \mathcal{P}(\mathtt{B}))|$, is an upper bound on the number of premisory and conclusory roles. However, Cantor's Theorem immediately implies that $|\mathtt{B}| < |\mathcal{P}(\mathtt{B}) \times \mathcal{P}(\mathtt{B})| < |\mathcal{P}(\mathbb{S})|$. Thus, the upper bounds on the number of conceptual contents defined by an implication frame is larger than the number of bearers in the implication frame. This holds only for the potential number of conceptual contents given by a set of bearers because there can be implication frames that do not define as many conceptual contents

as they have bearers. The trivial consequence relation over a collection of bearers, for example, defines only one single conceptual content.

8 If we wanted to think about conceptual contents as independent from particular implication frames, we could take another step of abstraction and say that conceptual contents defined by different implication spaces are equivalent if there is an entailment preserving bijection between the contents of the implication frames, perhaps as follows: Given two implication frames $\langle \mathtt{B}_1, \mathbb{I}_1 \rangle$ and $\langle \mathtt{B}_2, \mathbb{I}_2 \rangle$ and their contents $\mathbb{C}_1$ and $\mathbb{C}_2$, then the contents $\mathtt{a} \in \mathbb{C}_1$ and $\mathtt{b} \in \mathbb{C}_2$ are equivalent contents if and only if, there is a bijection, $f$, between subsets of $\mathbb{C}_1$ and $\mathbb{C}_2$ such that $\mathtt{P} \Vdash\!\sim \mathtt{C}$ holds in $\langle \mathtt{B}_1, \mathbb{I}_1 \rangle$ just in case $f(\mathtt{P}) \Vdash\!\sim f(\mathtt{C})$ holds in $\langle \mathtt{B}_2, \mathbb{I}_2 \rangle$ and $f(\mathtt{a}) = \mathtt{b}$.

With an equivalence relation among contents of different implication frames in hand, we could now define a notion of "absolute contents" as equivalence classes of contents with respect to this equivalence relation. However, we do not pursue this line of thought any further here. We do not pursue this thought because several issues arise at this point that we wish to set aside. For example, if there is more than one entailment preserving bijection between contents, then there are contents that are equivalent to two distinct contents. This happens if there are permutations of contents that preserve entailment. That might seem problematic. The issue is related to the question whether there are always several equally correct translations between any two languages (as Quine thought) and how we may be able to fix representation relations up to uniqueness, which we briefly touched upon in the previous chapter. In particular, when we consider bearers of implicational roles that are sentences of a language or vehicles of thought, the idea of Covariant Tracking might again be useful here. We may require, for example, that the bijection continues to hold under parallel additions of content bearers to both implication frames and that it continues to hold after parallel rational adjustments of the language in light of new evidence. Moreover, one might worry that our definition of equivalence of contents is not only too weak but also too strong because one might want to say that, for example, sentences in two languages have the same content although one of the two languages can express contents that the other language cannot express. Hence, one might choose to require a bijection between the contents of one implication frame and a subset of the contents of another implication frame. However, if the contents of the first implication frame are few and simple, then this relation might hold in cases in which we would intuitively deny that the contents are equivalent. Hence, the issue is a delicate one. We here set such difficult issues to one side again, as these issues do not seem specific to our account. Corresponding issues seem to arise in different ways for many theories of conceptual content and reason relations. And any tools that other theories might use to address them seem, in principle, available in our setting.

9 While languages have countably many sentences, there might be uncountably many worldly propositions. This is one respect in which the correspondence between truth-maker theory and implication-space semantics is even stronger than the correspondence between NMMS and implication-space semantics. However, we won't pursue such cardinality related issues any further here.

10 One might wonder how it can be that $[\![\phi]\!]^{\mathcal{M}} = |\phi|^{\mathcal{M}'}$ given that these objects can seem very different. Notice, however, that $[\![\phi]\!]^{\mathcal{M}} = \mathcal{R}(\phi) = \langle \mathcal{R}\langle\{\phi\},\emptyset\rangle, \mathcal{R}\langle\emptyset,\{\phi\}\rangle\rangle$, which is a pair of sets. And $|\phi|^{\mathcal{M}'} = \left\langle |\phi|^{+}, |\phi|^{-}\right\rangle$, which is also a pair of sets. So what the identity mentioned in this definition requires is that we let $\langle \mathcal{R}\langle\{\phi\},\emptyset\rangle, \mathcal{R}\langle\emptyset,\{\phi\}\rangle\rangle$ be $\left\langle |\phi|^{+}, |\phi|^{-}\right\rangle$.

11 Of course, the sentences interpreted by truth-maker models are also spaces of bearers of implicational roles. But it is more illuminating here to focus on worldly propositions.

12 The difference between contractive and non-contractive reason relations corresponds to the difference between monoids whose operation is idempotent and those whose operation is not idempotent.

13 Thinking in proof-theoretic terms, relaxing this constraint leads to so-called consecution calculi, in which we can encounter, for instance, tree structures of premises and conclusions. Such calculi are useful, for instance, in relevance logics and modal logics; but we will continue to ignore them here.

14 Daniel Kaplan (2022), who constructed the original implication-space semantics, in fact developed it by generalizing Girard's phase-space semantics because he was interested in reason relations that are nontransitive, nonmonotonic, and noncontractive.

15 In this and the next two paragraphs, we ignore some complications about the distinction between bearers and their roles. Taking them into account would make the explanation unnecessarily complicated and merely obscure what matters here.

16 To see why context-mixing Cut—given Containment—would preclude this, suppose that, for any $X$ and $Y$, if $X, A \mathrel{|\!\sim} Y$, then $X, B \mathrel{|\!\sim} Y$. By Containment, $A \mathrel{|\!\sim} A$ and, hence, $B \mathrel{|\!\sim} A$. Now suppose that $X \mathrel{|\!\sim} B, Y$. By context-mixing Cut, it follows from the last two sequents that $X \mathrel{|\!\sim} A, Y$. So, if context-mixing Cut and Containment hold, then substitutability as premises implies the converse substitutability as conclusions.

17 Daniel Kaplan has first suggested such a notion in our context, in a series of working-papers presented to the ROLE group. He connected it to three-valued logics, but he did not develop the idea in the context of implication-space semantics, as we do here.

18 The principles below concern how structural principles for the underlying implications show up in role inclusion relations. Regarding structural principles for role inclusion itself, Reflexivity holds because one can replace any role for itself.

An analogue of the rule of Weakening for the left side holds for role inclusion: If $\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m$, then $\mathcal{R}_0, \mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m$. This holds because $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i \sqcap \mathcal{R}_0) \subseteq \mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i)$, which is true because $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i \sqcap \mathcal{R}_0)$ is $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i) \cap \mathtt{RSR}(\mathcal{R}_0)$, which is obviously a subset of $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i)$. An analogue of the Weakening rule on the right side holds if the underlying implication relation is monotonic. That is, if the implicational relation is monotonic, then if $\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m$, then $\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m, \mathcal{R}_{m+1}$.

The following analogue of Cut holds for implicational role inclusion: if $\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_0$ and $\mathcal{R}_0, \mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m$, then $\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m$. For, the antecedent means that $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i) \subseteq \mathtt{RSR}(\mathcal{R}_0)$ and $\mathtt{RSR}(\mathcal{R}_0 \sqcap \bigsqcap_{i=1}^{n} \mathcal{R}_i) \subseteq \mathtt{RSR}(\bigsqcup_{j=k}^{m} \mathcal{R}_j)$. But, $\mathtt{RSR}(\mathcal{R}_0 \sqcap \bigsqcap_{i=1}^{n} \mathcal{R}_i) = \mathtt{RSR}(\mathcal{R}_0) \cap \mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i)$. Hence, $\mathtt{RSR}(\mathcal{R}_0) \cap \mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i) = \mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i)$. So $\mathtt{RSR}(\mathcal{R}_0 \sqcap \bigsqcap_{i=1}^{n} \mathcal{R}_i) = \mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i)$ and, thus, $\mathtt{RSR}(\bigsqcap_{i=1}^{n} \mathcal{R}_i) \subseteq \mathtt{RSR}(\bigsqcup_{j=k}^{m} \mathcal{R}_j)$. Therefore, $\mathcal{R}_1, ..., \mathcal{R}_n \preceq \mathcal{R}_k, ..., \mathcal{R}_m$.

19 From now on we use these semantic clauses; we assume Contraction by working with sets of bearers, and use set theoretic notions in their usual way again (for sets and not for multi-sets).

20 If we allowed ourselves to use open sentences and call their contents "concepts," we could, at this point, note that, plausibly, the concept of "_ is a dog" includes the concept of "_ is a mammal," thus vindicating the old idea that inclusions or containment relations among concepts often correspond to the inverse inclusions among the extensions of these concepts. Although working out this tantalizing idea goes beyond the scope of this book, having such an idea in mind can be helpful below when we turn to logics that have been suggested as logics for content.

21 These ascriptions would have to be qualified and explained in many respects to do justice to the positions of Priest or Kripke. We are here merely interested in the most straightforward and simplistic interpretation, as this works well enough for our current purposes.

22 This is related to work by Fitting (2021) on strict-tolerant logic and bilattices. One key difference, however, is that we do not use a lattice ordering to define consequence in implication-space models.

23 The connections that we point out in this subsection are closely related to the connections between these logics and admissible sequent rules (see Hlobil, 2022a).

24 Elgin's system is formulated in terms of predicates, but we can treat sentences as 0-place predicates. So this difference doesn't matter for our purposes. If we wanted the generality of Elgin's theory in our setting, we would need to include sub-sentential structure.

25 We formulate everything as double-line rules, which cuts down on clutter. What is new, relative to Tanter, are the rules G/FR, RFji, and LFji for the Species-Genus relation. These are variations on Tanter's CR and PR rules. We reject Tanter's CR-rule because in the absence of right-weakening the stipulation that if it follows that something belongs to a particular species, then it follows that it belongs to the genus is not equivalent with the stipulation that if it follows that something belongs to some species or other, then it follows that it belongs to the genus. We want to codify the latter and not the former. We reject Tanter's PR-rule because it is a formulation of the thought that something

belongs to a species if and only if it belongs to the genus and to none of the other species of that genus, which we can better express by saying that $F_j v$ should behave inferentially like $Gv \wedge \neg F_1 v \wedge \ldots \wedge \neg F_i v$, which is the same as $Gv \wedge \neg(F_1 v \vee \ldots \vee F_i v)$, which is equivalent to $\neg(\neg Gv \vee (F_1 v \vee \ldots \vee F_i v))$, which is equivalent to $\neg(Gv \to (F_1 v \vee \ldots \vee F_i v))$. And it is the latter version that is most easily seen in the sequent rules.

26 In particular, Correia uses the inclusive clauses and does not require bilateral propositions (our worldly propositions) to be convex, as Fine sometimes does.